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Uniqueness of best ϕ-approximation from the set of splines with infinitely many simple knots

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Abstract

Let J be an open interval and denote by $\mathscr{S}_\Pi$ the set of all the splines of degree at most $n-1$ with simple knots in Π, a countably infinite set of points in J, $n \geqslant 2$. In this paper, we prove that there exists a unique best ϕ-approximation to a continuous function in $\mathscr{L}_\phi(J)$ from $\mathscr{S}_\Pi$, where $\phi : [0, \infty) \mapsto [0, \infty)$ is a convex function that generalizes the pth-power functions, $p \geqslant 1$.

Keywords: Property A; Splines with infinitely many simple knots; Weak Chebyshev spaces; Uniqueness of best ϕ-approximation

1. Introduction

Throughout the paper ϕ will always denote any convex function defined on $[0, \infty)$, $\phi(0) = 0$ and $\phi(y) > 0$ for $y > 0$. Thus, ϕ has a right derivative at any point and a left derivative at any point in $(0, \infty)$, which we will denote by ϕ'_+ and ϕ'_-, respectively. We also assume that ϕ satisfies Property Δ_2, i.e., there exist $y_0 > 0$ and $c > 0$ such that $\phi(2y) \leqslant c\phi(y)$ for $y \geqslant y_0$. Under these conditions,

$$\mathscr{L}_\phi(K) := \left\{ h : h \text{ is Lebesgue measurable on } K \text{ and } \int_K \phi(|h|) < \infty \right\}$$

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is a linear space, where K is an arbitrary interval. For $\phi(y) = \phi_p(y) \equiv y^p$, $1 \leqslant p < \infty$, we write $L_p(K)$ instead of $\mathscr{L}_{\phi_p}(K)$.

Definition 1.1. Let $f \in \mathscr{L}_\phi(K)$, and let $\mathscr{M}$ be a set of functions defined on K. We say that g_0, in $\mathscr{M}$, is a best *ϕ-approximation to f from $\mathscr{M}$* if

$$\int_K \phi(|f - g_0|) \leqslant \int_K \phi(|f - g|) \quad \text{for all } g \in \mathscr{M}.$$

In the case $\phi = \phi_p$, $1 \leqslant p < \infty$, we will say that g_0 is a best L_p-approximation to f from $\mathscr{M}$.

It is well known that if $1 < p < \infty$, then there exists at most one best L_p-approximation when approximating from a convex set $\mathscr{M}$. This follows because the L_p-norm is strictly convex whenever $1 < p < \infty$. On the other hand, the uniqueness of best L_1-approximation does not follow a general rule and hence this is a question of main interest in the theory of best approximation. So a best L_1-approximation from a convex set may not be unique (for instance, see the example in [1]). However, provided f is continuous, uniqueness may be valid when approximating from some special classes of functions. For instance, Galkin [3] and Strauss [11] have showed that the problem of best L_1-approximation to a continuous function from the space of polynomial splines with finitely many fixed knots has a unique solution. Recently, in [2,5] we have given an affirmative answer to the problem of uniqueness of best L_1-approximation to a continuous $f \in L_1(J_0)$ from the set of n-convex functions, $n \geqslant 2$, J_0 being an open, bounded interval.

From now on, J will denote a fixed, open interval (a, b), $-\infty \leqslant a < b \leqslant +\infty$, and we will write $\mathscr{L}_\phi$ for $\mathscr{L}_\phi(J)$. Let Π be a set of infinitely many fixed points in J satisfying $\overline{\Pi} \cap J = \Pi$, i.e., $\Pi = \{x_i\}_{i \in \Gamma}$, with $a < x_i < x_j < b$ whenever $i < j$, where Γ is $\mathbb{Z}$, or $\mathbb{N}$, or $-\mathbb{N}$, and $x_i \to a$ as $i \to -\infty$, $x_j \to b$ as $j \to +\infty$. For a fixed and arbitrary integer $n \geqslant 2$ we will henceforth denote by $\mathscr{S}_\Pi$—its dependence on n is not indicated—the set of all piecewise polynomial functions g of degree at most $n-1$ with simple knots in Π (g is $(n-2)$ times continuously differentiable). Such splines arise naturally in the study of best ϕ-approximation to an $f \in \mathscr{L}_\phi$ from the set of n-convex functions (see [5,13]).

Our aim in this paper is to prove the existence and uniqueness of best ϕ-approximation to a continuous $f \in \mathscr{L}_\phi$ from $\mathscr{S}_\Pi$ for every ϕ with the conditions established at the beginning of the paper. In Section 2, we will prove the existence of best ϕ-approximation for any $f \in \mathscr{L}_\phi$, and in Section 3, after stating the so-called Property A, we will show that if $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfies Property A, then there exists a unique best ϕ-approximation to a continuous $f \in \mathscr{L}_\phi$ from $\mathscr{S}_\Pi$. Finally, in Section 4 we will prove that both $\mathscr{S}_\Pi$ and $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfy Property A.

We remark that the splines with infinitely many knots may be a useful tool in some applications. So, for instance, observe that if J is an unbounded interval then the unique best L_1-approximation to a continuous function from the set of splines with finitely many knots has a bounded support necessarily. On the other hand, the theory developed in the paper will allow us to show the existence of splines in L_1 with infinitely many knots and with unbounded support (see the example at the end of the paper).

Finally, we remark that in [2,5] we have proved that a proper subspace of $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfies Property A. In the present paper, we have used a similar technique to prove that $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfies Property A. But this similarity is formal only. Most of the main results in this paper have an independent difficulty and they cannot be solved as a consequence of the results obtained in [2,5].

2. Existence of a best ϕ-approximation from $\mathscr{S}_\Pi$

Theorem 2.1. *Let $f \in \mathscr{L}_\phi$. Then there exists a best ϕ-approximation to f from $\mathscr{S}_\Pi$.*

Proof. Recall that $J = (a, b)$, where $-\infty \leqslant a < b \leqslant +\infty$. We are assuming here that $\Pi = \{x_i\}_{i \in \mathbb{N}}$, i.e., $a < x_i < x_j < b$ whenever $i < j$, and $x_i \to b$ as $i \to +\infty$. Let

$$\gamma := \inf\left\{\int_J \phi(|f - g|) \colon g \in \mathscr{S}_\Pi\right\}.$$

Note that $\gamma < \infty$ since $0|_J$, the zero function on J, is in $\mathscr{S}_\Pi$, and $f \in \mathscr{L}_\phi$. For each $l \in \mathbb{N}$ we take a $g_l \in \mathscr{S}_\Pi$ satisfying

$$\int_J \phi(|f - g_l|) \leqslant \gamma + 1/l. \tag{1}$$

Note that in the case $a = -\infty$ each g_l vanishes identically on $(a, x_1]$. Assume now $a > -\infty$. Then we consider the Banach space $(\mathbb{P}_{n-1}[a, x_1], ||\cdot||_\phi)$, where $\mathbb{P}_{n-1}[a, x_1]$ is the set of the polynomials of degree at most $n-1$ restricted to $[a, x_1]$, and $||\cdot||_\phi$ is the Luxemburg norm, defined on $\mathscr{L}_\phi([a, x_1])$ by

$$||g||_\phi := \inf\left\{\lambda > 0 : \int_{[a,x_1]} \phi\left(\frac{|g|}{\lambda}\right) \leqslant 1\right\}.$$

As ϕ is an increasing, convex function, for each $l \in \mathbb{N}$ we have

$$\begin{aligned}\int_{[a,x_1]} \phi\left(\left|\frac{1}{2} g_l\right|\right) &\leqslant \int_{[a,x_1]} \phi\left(\frac{1}{2}|f - g_l| + \frac{1}{2}|f|\right)\\ &\leqslant \int_{[a,x_1]} \left(\frac{1}{2}\phi(|f - g_l|) + \frac{1}{2}\phi(|f|)\right) \leqslant M,\end{aligned}$$

where M is a constant, which we assume greater than 1, and the last inequality is justified because of (1) and the fact that f is in $\mathscr{L}_\phi$. Then

$$\int_{[a,x_1]} \phi\left(\frac{1}{2M}|g_l|\right) \leqslant \frac{1}{M} \int_{[a,x_1]} \phi\left(\frac{1}{2}|g_l|\right) \leqslant 1,$$

where the first inequality is due to the convexity of ϕ and to the fact that $\phi(0) = 0$. Thus, $||g_l||_\phi \leqslant 2M$. As $\mathbb{P}_{n-1}[a, x_1]$ is a finite-dimensional space, the ball

$$\{h \in \mathbb{P}_{n-1}[a, x_1] : ||h||_\phi \leqslant 2M\}$$

is a compact set. Therefore, there exist a polynomial $h_0 \in \mathbb{P}_{n-1}[a, x_1]$ and a subsequence $\{l_0\}_{l_0 \in \Lambda_0}$ of $\{l\}_{l \in \mathbb{N}}$ such that

$$||g_{l_0} - h_0||_\phi \to 0 \quad \text{as } l_0 \to +\infty.$$

So, by the equivalence of norms in the space $\mathbb{P}_{n-1}[a, x_1]$ we get

$$\{g_{l_0}\}_{l_0 \in \Lambda_0} \to h_0 \quad \text{uniformly on } (a, x_1] \text{ as } l_0 \to +\infty.$$

It is obvious that this last result is also valid in the case $a = -\infty$, with $\Lambda_0 = \mathbb{N}$.

Consider now the space $\mathbb{P}_{n-1}[x_1, x_2]$, endowed first with the Luxemburg norm. With the same procedure as before we deduce that there exist a subsequence $\{l_1\}_{l_1 \in \Lambda_1}$ of $\{l_0\}_{l_0 \in \Lambda_0}$ and a spline h_1, extension of h_0 on $(a, x_2]$ and with a simple knot at x_1, such that

$$\{g_{l_1}\}_{l_1 \in \Lambda_1} \to h_1 \quad \text{uniformly on } (a, x_2] \text{ as } l_1 \to +\infty.$$

In this way, for any $i \in \mathbb{N}$ we obtain a subsequence $\{l_i\}_{l_i \in \Lambda_i}$ of $\{l_{i-1}\}_{l_{i-1} \in \Lambda_{i-1}}$ and a spline h_i, extension of h_{i-1} on $(a, x_i]$ and with simple knots at the points $x_1, x_2, \ldots, x_{i-1}$, such that

$$\{g_{l_i}\}_{l_i \in \Lambda_i} \to h_i \quad \text{uniformly on } (a, x_i] \text{ as } l_i \to +\infty.$$

Thus, applying the Cantor diagonal procedure we get a subsequence $\{l\}_{l \in \Lambda}$ of $\{l\}_{l \in \mathbb{N}}$ and a $g_0 \in \mathscr{S}_\Pi$ such that

$$\{g_l\}_{l \in \Lambda} \to g_0 \quad \text{pointwise on } J \text{ as } l \to +\infty.$$

As ϕ is continuous, applying Fatou's lemma and taking into account (1) we deduce that

$$\gamma \leqslant \int_J \phi(|f - g_0|) = \int_J \lim_{\substack{l \to +\infty \\ l \in \Lambda}} \phi(|f - g_l|) \leqslant \liminf_{\substack{l \to +\infty \\ l \in \Lambda}} \int_J \phi(|f - g_l|) \leqslant \gamma.$$

Hence, it follows that g_0 is a best ϕ-approximation to f from $\mathscr{S}_\Pi$. If $\Gamma = -\mathbb{N}$, or $\Gamma = \mathbb{Z}$, the proof is analogous. □

3. Property A and uniqueness of best ϕ-approximation from $\mathscr{S}_\Pi$

The following lemma gives a characterization formula for a best ϕ-approximation from $\mathscr{S}_\Pi$ to a function $f \in \mathscr{L}_\phi$. Its proof is the same as that in [1, Lemma 1]. Observe that if g_0 is a best ϕ-approximation to $f \in \mathscr{L}_\phi$ from $\mathscr{S}_\Pi$, then g_0 is in $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$, since $f \in \mathscr{L}_\phi$, and $\mathscr{L}_\phi$ is a linear space.

Lemma 3.1. *Let $f \in \mathscr{L}_\phi$. The function g_0, in $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$, is a best ϕ-approximation to f from $\mathscr{S}_\Pi$ if, and only if, for all $h \in \mathscr{S}_\Pi \cap \mathscr{L}_\phi$ we have*

$$\int_J \mathscr{X}_{\{h>0\}} h(\mathscr{X}_{\{f \leqslant g_0\}} \phi'_+(|f-g_0|) - \mathscr{X}_{\{f>g_0\}} \phi'_-(|f-g_0|)) + \int_J \mathscr{X}_{\{h<0\}} h(\mathscr{X}_{\{f<g_0\}} \phi'_-(|f-g_0|) - \mathscr{X}_{\{f \geqslant g_0\}} \phi'_+(|f-g_0|)) \geqslant 0,$$

where $\mathscr{X}_R$ denotes the characteristic function of the set R.

Property A was introduced by Strauss [12] in an arbitrary finite-dimensional subspace of $\mathscr{C}(K_0)$, the set of the continuous functions on K_0, where K_0 is a compact interval, to provide a sufficient condition to ensure that there exists a unique best L_1-approximation to a function in $\mathscr{C}(K_0)$ from that subspace (Chapter 4 in [7] is devoted to study this property). We next establish Property A in any linear subspace $\mathscr{S}$ of $\mathscr{C}(K)$, where K is an arbitrary interval. Set

$$U^* := \{u^*, u^* \text{ is continuous on } K, |u^*| = |u| \text{ for some } u \in \mathscr{S} \backslash \{0\}\}.$$

Definition 3.2. We say that $\mathscr{S}$ satisfies *Property A* if to each $u^* \in U^*$ there exists an $h_0 \in \mathscr{S} \backslash \{0\}$ such that

(a) $h_0 = 0$ a.e. on $\{u^* = 0\}$; and
(b) $h_0 u^* \geqslant 0$ on J.

The following theorem provides us with a sufficient condition for uniqueness of best ϕ-approximation to a continuous function in $\mathscr{L}_\phi$ from $\mathscr{S}_\Pi$.

Theorem 3.3. *Let f be a continuous function in $\mathscr{L}_\phi$. If the space $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfies Property A, then there is a unique best ϕ-approximation to f from $\mathscr{S}_\Pi$.*

Proof. Let f be a continuous function in $\mathscr{L}_\phi$, and assume that g_0 and g_1, both in $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$, are two different best ϕ-approximations to f from $\mathscr{S}_\Pi$. Let $g_\lambda := (1-\lambda)g_0 + \lambda g_1$, $0 < \lambda < 1$. Using the convexity of ϕ we see immediately that for every λ the function g_λ is also a best ϕ-approximation to f from $\mathscr{S}_\Pi$. Using in addition the continuity of f we can assert that the sets $\{g_0 < f < g_1\}$ and $\{g_1 < f < g_0\}$ are empty. As a consequence, for $0 < \lambda_0 < \lambda_1 < 1$,

$$\{f > g_{\lambda_0}\} = \{f > g_{\lambda_1}\} \quad \text{and} \quad \{f < g_{\lambda_0}\} = \{f < g_{\lambda_1}\}.$$

Let $u := g_{\lambda_1} - g_{\lambda_0}$. We now define on J the function $u^* := |u| \operatorname{sgn}(f - g_{\lambda_0})$. So u^* is continuous, because u, f and g_{λ_0} are continuous, and $u = 0$ when $f = g_{\lambda_0}$. Moreover, $|u^*| = |u|$ and $u \in (\mathscr{S}_\Pi \cap \mathscr{L}_\phi) \backslash \{0\}$. Furthermore,

$$u^* = 0 \quad \text{on } \{f = g_{\lambda_0}\},$$

$$u^* \geqslant 0 \quad \text{on } \{f > g_{\lambda_0}\},$$

$$u^* \leqslant 0 \quad \text{on } \{f < g_{\lambda_0}\}.$$

If $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfies Property A, then there exists an $h_0 \in (\mathscr{S}_\Pi \cap \mathscr{L}_\phi)\backslash\{0\}$ such that (a) and (b) in Definition 3.2 hold with this function u^* and with $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ in the place of $\mathscr{S}$. Hence,

$$h_0 = 0 \quad \text{a.e. on } \{f = g_{\lambda_0}\},$$

$$h_0 \geqslant 0 \quad \text{on } \{f > g_{\lambda_0}\},$$

$$h_0 \leqslant 0 \quad \text{on } \{f < g_{\lambda_0}\}.$$

Thus, the characterization formula in Lemma 3.1 fails when g_0 is replaced by g_{λ_0}, and h by h_0, which contradicts that g_{λ_0} is a best ϕ-approximation to f from $\mathscr{S}_\Pi$. □

In the following section, we will prove that both $\mathscr{S}_\Pi$ and $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ indeed satisfy Property A. When J is a bounded interval it is known that the space of the splines with finitely many simple knots satisfies Property A [11]. We remark that for $n \geqslant 3$ the difficulty of the proof that $\mathscr{S}_\Pi$ ($\mathscr{S}_\Pi \cap \mathscr{L}_\phi$) satisfies Property A lies in the existence of splines in $\mathscr{S}_\Pi$ (in $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$) with infinitely many isolated zeros. (We show such a spline in $\mathscr{S}_\Pi \cap L_1$ in the example at the end of the paper).

4. Property A in the spaces $\mathscr{S}_\Pi$ and $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$

4.1. *Definitions and first results*

In the following definitions K denotes an interval, and h is a function in $\mathscr{C}(K)$.

Definition 4.1. We say that h has r *sign changes* if there exists $c_0 < c_1 < \cdots < c_{r+1}$ in K satisfying $h(c_i)h(c_{i+1}) < 0$, $i = 0, 1, \ldots, r$, and there exists no set $c'_0 < c'_1 < \cdots < c'_{r'+1}$ in K satisfying that property with $r' > r$. If $h|_L$, the restriction of h to an interval $L \subseteq K$, has r sign changes, then we also say that h has r *sign changes on L*.

Definition 4.2. For $m > 0$, an m-dimensional linear subspace $\mathscr{T}$ of $\mathscr{C}(K)$ is said to be *weak Chebyshev* (WT-space) if every function in $\mathscr{T}$ has at most $m-1$ sign changes.

Definition 4.3. Assume that (a_1, b_1) is the interior set of K, $-\infty \leqslant a_1 < b_1 \leqslant +\infty$. For an integer $m > 0$ we say that the set F, $\mathrm{Card}(F) = m-1$, is a (finite) *alternating set for h* if $F = \{z_l\}_{l=1}^{m-1}$, $a_1 < z_1 < z_2 < \cdots < z_{m-1} < b_1$, and either $(-1)^l h \geqslant 0$ on (z_l, z_{l+1}), $l = 0, 1, \ldots, m-1$, or $(-1)^l h \leqslant 0$ on (z_l, z_{l+1}), $l = 0, 1, \ldots, m-1$, where $z_0 = a_1$ and $z_m = b_1$. This definition, for a finite set F, has a natural generalization to a set $\Omega = \{z_l\}_{l\in\mathbb{Z}}$ of infinitely many isolated points in (a_1, b_1) satisfying $z_{l_1} < z_{l_2}$ whenever $l_1 < l_2$, and $z_l \to a_1$ as $l \to -\infty$, $z_l \to b_1$ as $l \to +\infty$. Indeed, we say that Ω is an (infinite) alternating set for h if either $(-1)^l h \geqslant 0$ on (z_l, z_{l+1}), all $l \in \mathbb{Z}$, or $(-1)^l h \leqslant 0$ on (z_l, z_{l+1}), all $l \in \mathbb{Z}$.

An m-dimensional WT-subspace $\mathscr{T}$ of $\mathscr{C}[a_1, b_1]$, $-\infty < a_1 < b_1 < +\infty$, satisfies the following two properties (see [9] or [10] for the first; [4] or [14], Lemma 4.1, for the second):

[SSS] For $m>1$, $\mathscr{T}$ contains an $(m-1)$-dimensional WT-subspace.

[JKZ] For $m>0$, given a set $F\subset(a_1,b_1)$, $\mathrm{Card}(F)=m-1$, there exists a function in $\mathscr{T}\backslash\{0\}$ for which F is an alternating set.

Definition 4.4. An interval $L\subseteq K$ is said to be a *zero interval* of h if $\mathrm{Card}(L)>1$ and h vanishes identically on L. The zero interval L is a *maximal* zero interval of h if L is not strictly contained in any zero interval of h.

Definition 4.5. A point $z\in K$ is a *zero of h* if $h(z)=0$; z is an *isolated* zero of h if there exists a $\delta>0$ such that z is the unique zero of h in $K\cap(z-\delta,z+\delta)$.

Definition 4.6. Let z be an isolated zero of h in $(a_1,b_1)\subseteq K$. We say that z is a *simple* zero of h if h changes sign at z. The point z is a *double* zero of h if h does not change sign at z.

Definition 4.7. Given an interval $L\subseteq K$, suppose that $h|_L$, the restriction of h to L, has finitely many isolated zeros, as well as finitely many sign changes on L. Then the number of isolated zeros of $h|_L$ shall be denoted by $Z_L(h)$, and $Z_L^2(h)$ will denote the number of sign changes of $h|_L$ plus twice the number of double zeros of $h|_L$ plus the number of endpoints of L that in addition are isolated zeros of $h|_L$.

The following result follows from Lemma 3(a) in [5].

Lemma 4.8. *Let $h\in\mathscr{C}(K)$, where $K=[a_1,b_1]$, and let $I[h](x) := \int_{a_1}^{x} h$, all $x\in[a_1,b_1]$. Assume that h has at most r sign changes. Then the function $c+I[h]$ has finitely many sign changes and finitely many isolated zeros for every constant c. More precisely, $Z^2_{[a_1,b_1]}(c+I[h])\leqslant r+1$, whence $c+I[h]$ has at most $r+1$ sign changes.*

Definition 4.9. For a fixed and arbitrary integer $n\geqslant 2$, we denote by $\mathscr{S}_{i,j}$—its dependence on n is not indicated—the linear space of the restrictions to $[x_i,x_j]$ of all functions in $\mathscr{S}_\Pi$. Furthermore, $\mathscr{S}^+_{i,j}$ ($\mathscr{S}^-_{i,j}$) will denote the linear subspace of $\mathscr{S}_{i,j}$ that consists of all the splines $G\in\mathscr{S}_{i,j}$ satisfying $G_+^{(l)}(x_i)=0$ ($G_-^{(l)}(x_j)=0$, respectively) for $l=0,1,\dots,n-2$. Finally, $\mathscr{S}^0_{i,j} := \mathscr{S}^+_{i,j}\cap\mathscr{S}^-_{i,j}$.

The sets $\mathscr{S}_{i,j}$, $\mathscr{S}^+_{i,j}$, $\mathscr{S}^-_{i,j}$ and $\mathscr{S}^0_{i,j}$ are WT-spaces (cf. [6]), and because of Properties [SSS] and [JKZ] it can be proved that they also satisfy Property A [12]. We have

$$\dim\mathscr{S}_{i,j}=j-i+n-1,$$

$$\dim\mathscr{S}^+_{i,j}=\dim\mathscr{S}^-_{i,j}=j-i,$$

$$\dim\mathscr{S}^0_{i,j}=\max\{0,j-i-n+1\}.$$

Let g be a function in $\mathscr{S}_\Pi$, or in $\mathscr{S}_{i',j'}$, $i'\leqslant i<j\leqslant j'$. We write $Z_{[i,j]}(g)$ for $Z_{[x_i,x_j]}(g)$, and mutatis mutandis for $Z^2_{[x_i,x_j]}(g)$, as well as for (x_i,x_j), $[x_i,x_j)$ and $(x_i,x_j]$.

Definition 4.10. For $h \in \mathscr{S}_{i,j} \setminus \{0\}$ we denote by $Z^{\star}_{[i,j]}(h)$ the number of zeros of h counting multiplicities, such as in [8, Section 4.7] (zero intervals also count in $Z^{\star}_{[i,j]}(h)$). We have $Z_{[i,j]}(h) \leqslant Z^{2}_{[i,j]}(h) \leqslant Z^{\star}_{[i,j]}(h)$.

Lemma 4.11. *Let g be in $\mathscr{S}_{\Pi}$, or in $\mathscr{S}_{i',j'}$, $i' \leqslant i < j \leqslant j'$, where g does not vanish identically on $[x_i, x_j]$. Then*

(a) $Z^{\star}_{[i,j]}(g) \leqslant j - i + n - 2$.
(b) *If the restriction of g to $[x_i, x_j]$ is in $\mathscr{S}^{+}_{i,j}$, or in $\mathscr{S}^{-}_{i,j}$, then $Z_{(i,j]}(g) \leqslant j - i - 1$, or $Z_{[i,j)}(g) \leqslant j - i - 1$, respectively.*
(c) *If the restriction of g to $[x_i, x_j]$ is in $\mathscr{S}^{0}_{i,j}$, then $Z_{(i,j)}(g) \leqslant j - i - n$.*

Proof. Item (a) follows from [8, Theorem 4.53]. To prove (b) and (c), suppose that $g|_{[x_i,x_j]}$ is in $\mathscr{S}^{+}_{i,j}$. If x_i is an isolated zero of $g|_{[x_i,x_j]}$, then the multiplicity of x_i is $n-1$. Otherwise the multiplicity of the zero interval $[x_i, x_l]$, $i < l < j$, is not smaller than n. Hence, $Z_{(i,j]}(g) \leqslant Z^{\star}_{[i,j]}(g) - (n-1)$. A similar rule holds when $g|_{[x_i,x_j]}$ is in $\mathscr{S}^{-}_{i,j}$. Now (b) and (c) follow from (a). $\square$

In order to prove the uniqueness of best ϕ-approximation to a continuous function in $\mathscr{L}_{\phi}$ from $\mathscr{S}_{\Pi}$, and according to Theorem 3.3, our aim is now to demonstrate that the space $\mathscr{S}_{\Pi} \cap \mathscr{L}_{\phi}$ indeed satisfies Property A. The following theorem states that both $\mathscr{S}_{\Pi}$ and $\mathscr{S}_{\Pi} \cap \mathscr{L}_{\phi}$ satisfy Property A.

Theorem 4.12. *Let u be in $\mathscr{S}_{\Pi} \setminus \{0\}$, and let u^* denote a continuous function defined on J satisfying $|u^*| = |u|$. Then there exists an $h_0 \in \mathscr{S}_{\Pi} \setminus \{0\}$ such that*

(a) $h_0 = 0$ *a.e. on* $\{u^* = 0\}$;
(b) $h_0 u^* \geqslant 0$ *on* J; *and*
(c) *If u is in $\mathscr{L}_{\phi}$, then so is h_0.*

It follows from Theorem A, in [2,5], the existence of an $h_0 \in \mathscr{S}_{\Pi}$ (for a bounded J) satisfying (a) and (b) of Theorem 4.12. However, we are not able to determine whether h_0 satisfies (c) as well. So, even in the case where J is bounded, we cannot make use of the previous existence of that theorem to prove Theorem 4.12, except for the fact that we will use an analogous technique to that employed in the proof of Theorem A. In fact, both proofs are based on the construction of appropriate weak Chebyshev spaces defined in terms of *levels* (Definition 4.27). But we emphasize that in this paper the definition of level is not the same as that employed in Theorem A.

Theorem 4.12 will be finally proved in Section 4.3. We first need several results. We begin with the following two lemmas, which allows us to reduce the number of cases to consider, depending on the form of Γ and u. We omit their proofs because they are completely similar to those of Lemmas 8 and 9 in [5], respectively.

Lemma 4.13. *If Theorem* 4.12 *holds when* $\Gamma = \mathbb{Z}$, *then it also holds when* $\Gamma = \mathbb{N}$, *or* $\Gamma = -\mathbb{N}$.

Lemma 4.14. *Theorem* 4.12 *holds when* $\Gamma = \mathbb{Z}$ *and u has at least two maximal zero intervals.*

According to Lemmas 4.13 and 4.14, we henceforth suppose that $\Gamma = \mathbb{Z}$ and that the function u, in the statement of Theorem 4.12, has at most one maximal zero interval. However, the main case to consider is that in which u has no zero interval. Under this condition the zeros of u are isolated, and therefore the continuous function u^*, defined on J and satisfying $|u^*| = |u|$, determines a set of isolated points in J, namely

$$\Omega := \{z \in J : z \text{ is a simple zero of } u^*\}.$$

For $i < j$, set

$$\Omega_{(i,j)} := \operatorname{Card}(\Omega \cap (x_i, x_j)), \quad \Omega_{[i,j]} := \operatorname{Card}(\Omega \cap [x_i, x_j]),$$
$$\Omega_{[i,j)} := \operatorname{Card}(\Omega \cap [x_i, x_j)), \quad \Omega_{(i,j]} := \operatorname{Card}(\Omega \cap (x_i, x_j]).$$

Lemma 4.15. *Theorem* 4.12 *holds when* $\Gamma = \mathbb{Z}$, *the function u has no zero interval and* u^* *is such that* $\Omega_{(i,j)} < j - i - n + 1$ *for some* $i < j$.

Proof. If $\Omega_{(i,j)} < j - i - n + 1$ for some $i < j$, then we can use [JKZ] in an $(\Omega_{(i,j)} + 1)$-dimensional WT-subspace of $\mathscr{S}^0_{i,j}$, after applying [SSS] if $\Omega_{(i,j)} < j - i - n$, to show that there exists a spline $H_0 \in \mathscr{S}^0_{i,j} \backslash \{0\}$ for which $\Omega \cap (x_i, x_j)$ is an alternating set. Therefore, we define an h_0 in $\mathscr{S}_\Pi \backslash \{0\}$ by

$$h_0(x) := \begin{cases} H_0(x), & x \in [x_i, x_j], \\ 0, & x \in J \backslash [x_i, x_j]. \end{cases}$$

Then it is clear that either h_0 or $-h_0$ satisfies (a)–(c) of Theorem 4.12. □

According to Lemma 4.15, we now establish the following definition.

Definition 4.16. We say that $\{u, \Omega\}$ is a *reference pair* if u is in $\mathscr{S}_\Pi$ without zero intervals, where $\Pi = \{x_i\}_{i \in \mathbb{Z}}$, and Ω is a set of zeros of u satisfying

$$\Omega_{(i,j)} \geqslant j - i - n + 1 \quad \text{for any } i < j.$$

Proposition 4.17. *Let* $\{u, \Omega\}$ *be a reference pair. Then there exist two sequences of integers* $\{\imath_v\}_{v \in \mathbb{N}}$ *and* $\{\jmath_v\}_{v \in \mathbb{N}}$ *satisfying*

$$\cdots < \imath_v < \cdots < \imath_2 < \imath_1 \leqslant -n < n \leqslant \jmath_1 < \jmath_2 \cdots < \jmath_v < \cdots \tag{2}$$

and the following properties:

$$\{u=0\}\cap(J\backslash(x_{\imath_1},x_{\jmath_1}))=\Omega\cap(J\backslash(x_{\imath_1},x_{\jmath_1})), \tag{3}$$

$$\begin{cases}\Omega_{(j,\jmath_{v+1}]}\geqslant \jmath_{v+1}-j & \textit{for all } v\in\mathbb{N} \quad \textit{and any } \jmath_v\leqslant j<\jmath_{v+1},\\ \Omega_{[\imath_{v+1},i)}\geqslant i-\imath_{v+1} & \textit{for all } v\in\mathbb{N} \textit{ and any } \imath_{v+1}<i\leqslant \imath_v,\end{cases} \tag{4}$$

$$\begin{cases}\Omega_{(\jmath_v,\jmath_\mu]}=\jmath_\mu-\jmath_v & \textit{for any } v<\mu,\\ \Omega_{[\imath_\mu,\imath_v)}=\imath_v-\imath_\mu & \textit{for any } v<\mu.\end{cases} \tag{5}$$

Proof. Applying Lemma 4.11(a) to the function u, we have

$$Z^{\star}_{[i,j]}(u)\leqslant j-i+n-2.$$

Hence, if $\{u,\Omega\}$ is a reference pair then

$$j-i-n+1\leqslant\Omega_{(i,j)}\leqslant\Omega_{[i,j]}\leqslant Z_{[i,j]}(u)\leqslant Z^{\star}_{[i,j]}(u)\leqslant j-i+n-2 \tag{6}$$

for any $i<j$. So, in particular, if $i=0$ then for every $j\in\mathbb{N}$ we get

$$1-n\leqslant\Omega_{[0,j]}-j\leqslant Z_{[0,j]}(u)-j\leqslant n-2.$$

Then $\Omega\cap[x_{j'},b)=\{u=0\}\cap[x_{j'},b)$ for j' sufficiently large, since $Z_{[0,j]}(u)-\Omega_{[0,j]}\leqslant 2n-3$ for all $j\in\mathbb{N}$. In addition, since $\{\Omega_{[0,j]}-j\}_{j\in\mathbb{N}}$ is bounded it is easily checked that there exists a sequence of integers $(n\leqslant)\jmath_1<\jmath_2<\cdots<\jmath_v<\cdots$, such that

$$\Omega_{[0,\jmath_v]}-\jmath_v=\limsup_{j\to+\infty}(\Omega_{[0,j]}-j),\quad \text{all } v\in\mathbb{N},$$

$$\{u=0\}\cap[x_{\jmath_1},b)=\Omega\cap[x_{\jmath_1},b),$$

and

$$\Omega_{[0,j]}-j\leqslant\Omega_{[0,\jmath_v]}-\jmath_v \quad \text{for all } v\in\mathbb{N} \text{ and } j>\jmath_1.$$

Thus, it is clear that

$$\Omega_{(j,\jmath_{v+1}]}=\Omega_{[0,\jmath_{v+1}]}-\Omega_{[0,j]}\geqslant\jmath_{v+1}-j \quad \text{for all } v\in\mathbb{N} \text{ and any } \jmath_v\leqslant j<\jmath_{v+1},$$

and

$$\Omega_{(\jmath_v,\jmath_\mu]}=\Omega_{[0,\jmath_\mu]}-\Omega_{[0,\jmath_v]}=\jmath_\mu-\jmath_v \quad \text{for any } v<\mu.$$

Taking now $j=0$ in (6), and reasoning in a similar way, we conclude the proof. □

Definition 4.18. Whenever $\{u,\Omega\}$ is a reference pair, set

$$\mathscr{S}_v:=\{G\in\mathscr{S}_{\imath_v,\jmath_v}:\{G=0\}\cap\{x_{\imath_v},x_{\jmath_v}\}\supseteq\Omega\cap\{x_{\imath_v},x_{\jmath_v}\}\}.$$

Note that $\mathscr{S}_v$ is a linear space satisfying $\mathscr{S}^0_{\imath_v,\jmath_v} \subseteq \mathscr{S}_v \subseteq \mathscr{S}_{\imath_v,\jmath_v}$, with $\dim \mathscr{S}_v = \jmath_v - \imath_v + n - 1 - \mathrm{Card}(\Omega \cap \{x_{\imath_v}, x_{\jmath_v}\})$. Moreover, $\mathscr{S}_v$ is a WT-space (cf. [6]). We will henceforth write

$$m_v := \dim \mathscr{S}_v = \jmath_v - \imath_v + n - 1 - \sigma(\imath_v) - \sigma(\jmath_v),$$

where

$$\sigma(\imath_v) := \mathrm{Card}(\Omega \cap \{x_{\imath_v}\}) \quad \text{and} \quad \sigma(\jmath_v) := \mathrm{Card}(\Omega \cap \{x_{\jmath_v}\}).$$

Furthermore,

$$\imath_v^+ := \imath_v + \sigma(\imath_v) \quad \text{and} \quad \jmath_v^- := \jmath_v - \sigma(\jmath_v).$$

The following theorem follows from Theorem 2 in [5]. So we omit the proof, which is based on properties (4) and (5).

Theorem 4.19. *Let $\{u, \Omega\}$ be a reference pair, and let $G \in \mathscr{S}_1$. Then there exists a unique spline g in $\mathscr{S}_\Pi$, which we will denote by $\Psi(G)$, such that $\Psi(G) = G$ on $[x_{\imath_1}, x_{\jmath_1}]$ and*

$$\{\Psi(G) = 0\} \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})) \supseteq \Omega \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})). \tag{7}$$

Moreover, Ψ determines an isomorphism between $\mathscr{S}_1$ and $\Psi(\mathscr{S}_1) := \{\Psi(G): G \in \mathscr{S}_1\}$.

We are now in a position to prove Theorem 4.12 for the case $n = 2$. We will later obtain some results which are valid only if $n \geqslant 3$.

Theorem 4.20. *Theorem 4.12 holds for the case $n = 2$.*

Proof. According to Lemmas 4.13 and 4.14, we assume $\Gamma = \mathbb{Z}$ and $u \in \mathscr{S}_\Pi \backslash \{0\}$ has at most one maximal zero interval.

Suppose first that u has no zero interval. From Lemma 4.15 it is sufficient to consider the case in which $\{u, \Omega\}$ is a reference pair. Due to (3) and (5), $Z_{(\jmath_v, \jmath_{v+1}]}(u) = \Omega_{(\jmath_v, \jmath_{v+1}]} = \jmath_{v+1} - \jmath_v$ and $Z_{[\imath_{v+1}, \imath_v)}(u) = \Omega_{[\imath_{v+1}, \imath_v)} = \imath_v - \imath_{v+1}$ for every $v \geqslant 1$, and this implies that the continuous *broken line* u has one simple zero in (x_j, x_{j+1}) for $j \geqslant \jmath_1$, as well as one simple zero in (x_{i-1}, x_i) for $i \leqslant \imath_1$. Hence, $u(x_{\jmath_v}) \neq 0$ and $u(x_{\imath_v}) \neq 0$ for every $v \in \mathbb{N}$. Thus, $\mathscr{S}_1 = \mathscr{S}_{\imath_1, \jmath_1}$ and $\dim \mathscr{S}_1 = \jmath_1 - \imath_1 + 1$. After applying [SSS] if $\Omega_{(\imath_1, \jmath_1)} < \jmath_1 - \imath_1$, we use [JKZ] in an $(\Omega_{(\imath_1, \jmath_1)} + 1)$-dimensional WT-subspace of $\mathscr{S}_1$ to obtain an $H_0 \in \mathscr{S}_1 \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{\jmath_1})$ is an alternating set. Now apply Theorem 4.19 to H_0 and let $h_0 := \Psi(H_0)$. By (7) and (3),

$$\{h_0 = 0\} \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})) \supseteq \Omega \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})) = \{u = 0\} \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})).$$

Then since h_0 is a broken line, it follows that the restriction of h_0 to $[x_{\jmath_1}, b)$ is of the form $c_1 u$, where $|c_1| = |H_0(x_{\jmath_1})/u(x_{\jmath_1})|$, and analogously, the restriction of h_0 to $(a, x_{\imath_1}]$ is of the form $c_1' u$, where $|c_1'| = |H_0(x_{\imath_1})/u(x_{\imath_1})|$. Accordingly, (c) in Theorem 4.12 holds. Furthermore, the only (simple) zeros of h_0 in $J \backslash (x_{\imath_1}, x_{\jmath_1})$ are the points in $\Omega \cap (J \backslash (x_{\imath_1}, x_{\jmath_1}))$. Thus, Ω becomes an alternating set for h_0, i.e., (a) and (b) in

Theorem 4.12 hold as well. So, for $n=2$, Theorem 4.12 is true when u has no zero interval.

Suppose now that $u \in \mathscr{S}_\Pi \backslash \{0\}$ has only one maximal zero interval. Using considerations of symmetry, we can assume that this zero interval is not $[x_j, b)$ for any $j \in \mathbb{Z}$. Without loss of generality, suppose $[x_{-1}, x_0]$ is a zero interval of u and $u(x_1) \neq 0$. If $\operatorname{Card}(\Omega \cap (x_0, x_j)) < j-1$ for some $j > 1$, then apply Property A in the $(j-1)$-dimensional space $\mathscr{S}^0_{0,j}$ to the restrictions to $[x_0, x_j]$ of u and u^*. In this way the function which results from this application is trivially extended to J to show that Theorem 4.12 holds in this case. Thus, consider $\operatorname{Card}(\Omega \cap (x_0, x_j)) = j-1$ for every $j \geqslant 1$. Then u^* and u have the same (simple) zeros in (x_0, b), whence either $u^* = u$ or $u^* = -u$ on $[x_0, b)$. Finally, taking $h_0 = 0$ on (a, x_0) and $h_0 = u^*$ on $[x_0, b)$, it is easy to see that h_0 satisfies (a)–(c) of Theorem 4.12. □

In what follows, and until the final proof of Theorem 4.12 in Section 4.3, we will work under the assumption that $\{u, \Omega\}$ is a reference pair (Definition 4.18). In this way, we shall deal with the knots $x_{\imath_\nu}$, $x_{\jmath_\nu}$ and the WT-spaces $\mathscr{S}_\nu$, $\nu \in \mathbb{N}$, and with the linear map Ψ. Note that $\bigcup_{\nu=1}^{\infty} (x_{\imath_\nu}, x_{\jmath_\nu}) = J$.

Lemma 4.21. *For $n \geqslant 3$, let $G \in \mathscr{S}_1$ and assume that for some $\mu \in \mathbb{N}$ the restriction of $g := \Psi(G)$ to $(x_{\imath_\mu}, x_{\jmath_\mu})$ has $m_\mu - 1$ sign changes. Then g has no zero interval, all the zeros of g have multiplicity one, and*

$$\{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{\jmath_\mu})) = \Omega \cap (J \backslash (x_{\imath_\mu}, x_{\jmath_\mu})).$$

Moreover, $g^{(n-2)}$ has a simple zero in each (x_i, x_{i+1}), all $i \in \mathbb{Z}$.

Proof. We first prove that g has no zero interval. If G has a zero interval, then using the definition of multiplicity of a zero interval (see [8, Section 4.7]) we get $Z^*_{[\imath_\mu, \jmath_\mu]}(G) \geqslant m_\mu - 1 + \sigma(\jmath_\mu) + \sigma(\imath_\mu) + n - 1 = \jmath_\mu - \imath_\mu + n - 2 + n - 1$, which contradicts Lemma 4.11(a).

Assume now that g has a zero interval $[x_{j'}, x_{j'+1}]$, $j' \geqslant \jmath_\nu$, and that g has no zero interval in $[x_{\imath_\mu}, x_{j'}]$. Observe first that $j' = \jmath_\nu$ is not possible. Indeed, in this case the restriction of g to $(x_{\imath_\mu}, x_{\jmath_\mu})$ is in the $(\jmath_\mu - \imath_\mu - \sigma(\imath_\mu))$-dimensional WT-space $\mathscr{S}^-_{\imath_\mu, \jmath_\mu} \cap \mathscr{S}_\mu$. On the other hand, by hypothesis g has $m_\mu - 1$ sign changes on $(x_{\imath_\mu}, x_{\jmath_\mu})$, and

$$m_\mu - 1 = \jmath_\mu - \imath_\mu + n - 2 - \sigma(\imath_\mu) - \sigma(\jmath_\mu) \geqslant \jmath_\mu - \imath_\mu - \sigma(\imath_\mu),$$

since $n \geqslant 3$. This is a contradiction. Thus, $j' > \jmath_\mu$ necessarily. Let $\jmath_{\nu_0} \in \mathbb{Z}$ satisfying $\jmath_{\nu_0} > j'$. Note first that

$$\Omega_{[j', \jmath_{\nu_0}]} = Z_{[j', \jmath_{\nu_0}]}(u) \leqslant Z^*_{[j', \jmath_{\nu_0}]}(u) \leqslant \jmath_{\nu_0} - j' + n - 2,$$

where the equality is due to (3), and the last inequality follows from Lemma 4.11(a). Then, taking into account (5),

$$\Omega_{(J_\mu, j')} = \Omega_{(J_\mu, J_{v_0}]} - \Omega_{[j', J_{v_0}]} \geqslant J_{v_0} - J_\mu - (J_{v_0} - j' + n - 2) = j' - J_\mu - n + 2.$$

Since g has $m_\mu - 1$ sign changes on $(x_{\imath_\mu}, x_{J_\mu})$ and has no zero interval in $[x_{\imath_\mu}, x_{j'}]$, it follows that

$$Z_{(\imath_\mu, J_\mu)}(g) \geqslant m_\mu - 1.$$

Thus,

$$\begin{aligned} Z_{[\imath_\mu, j')}(g) &= Z_{[\imath_\mu, J_\mu]}(g) + Z_{(J_\mu, j')}(g) \geqslant Z_{(\imath_\mu, J_\mu)}(g) + \sigma(\imath_\mu) + \sigma(J_\mu) + \Omega_{(J_\mu, j')} \\ &\geqslant m_\mu - 1 + \sigma(\imath_\mu) + \sigma(J_\mu) + j' - J_\mu - n + 2 = j' - \imath_\mu, \end{aligned}$$

where (7) is used in the first inequality. So $Z_{[\imath_\mu, j')}(g) \geqslant j' - \imath_\mu$, which contradicts Lemma 4.11(b) since the restriction of g to $[x_{\imath_\mu}, x_{j'}]$ is in $\mathscr{S}^-_{\imath_\mu, j'}$. In consequence, g has no zero interval in $[x_{J_\mu}, b)$. The proof that g has no zero interval in $(a, x_{\imath_\mu}]$ is completely symmetrical. Therefore, g has no zero interval.

To prove that all the zeros of g have multiplicity one, and that $\{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$, consider an arbitrary integer $v > \mu$. As g has no zero interval, we get

$$\begin{aligned} Z^*_{[\imath_v, J_v]}(g) &\geqslant Z_{[\imath_v, J_v]}(g) = Z_{[\imath_\mu, J_\mu]}(g) + Z_{[\imath_v, \imath_\mu)}(g) + Z_{(J_\mu, J_v]}(g) \\ &\geqslant m_\mu - 1 + \sigma(\imath_\mu) + \sigma(J_\mu) + \Omega_{[\imath_v, \imath_\mu)} + \Omega_{(J_\mu, J_v]} \\ &= J_\mu - \imath_\mu + n - 2 + \imath_\mu - \imath_v + J_v - J_\mu = J_v - \imath_v + n - 2, \end{aligned} \tag{8}$$

where (7) is used in the second inequality, and the second equality follows from (5).

If z is a zero of g of multiplicity greater than one, then the first inequality in (8) is strict for all $v > \mu$ with the property that $z \in (x_{\imath_v}, x_{J_v})$. Therefore, we obtain a contradiction since from Lemma 4.11(a), $Z^*_{[\imath_v, J_v]}(g) \leqslant J_v - \imath_v + n - 2$. Thus, all the zeros of g have multiplicity one.

Due to (7), $\{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) \supseteq \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$. Hence, to prove that $\{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$ it is sufficient to see that

$$\{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) \subseteq \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})).$$

Suppose that there exists a point $z' \in \{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$ such that $z' \notin \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$. Then the second inequality in (8) is strict for all $v > \mu$ with the property that $z' \in (x_{\imath_v}, x_{J_v})$. In this way, we again obtain a contradiction. Thus, $\{g = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$.

We finally prove that $g^{(n-2)}$ has a simple zero in each (x_i, x_{i+1}), all $i \in \mathbb{Z}$. From (8) and Lemma 4.11(a) it follows that $Z^*_{[\imath_v, J_v]}(g) = Z_{[\imath_v, J_v]}(g) = J_v - \imath_v + n - 2$ for any

$\nu > \mu$. Therefore, $Z^2_{[\imath_\nu, \jmath_\nu]}(g) = \jmath_\nu - \imath_\nu + n - 2$ for any $\nu > \mu$. Then applying $n-2$ times Lemma 4.8 we deduce that $g^{(n-2)}$ has $\jmath_\nu - \imath_\nu$ sign changes on $[x_{\imath_\nu}, x_{\jmath_\nu}]$. This means that the broken line $g^{(n-2)}$ has a simple zero in each (x_i, x_{i+1}), all $i \in \mathbb{Z}$, since ν is an arbitrary integer greater than μ. $\square$

Definition 4.22. Let $G \in \mathscr{S}_1$. We say that $g := \Psi(G)$ *goes to* 0 *to the right* if $g = 0$ on (x_j, b) for some $j \in \mathbb{Z}$, and g *goes to* 0 *to the left* if $g = 0$ on (a, x_i) for some $i \in \mathbb{Z}$. It is said that g *goes to* 0 if g goes to 0 to the right and to the left.

Lemma 4.23. *Let $g := \Psi(G)$, where $G \in \mathscr{S}_1$. Then there holds*:

(a) *If g does not go to* 0 *to the right, then g has no zero interval in $[x_{\jmath_1 - 1}, b)$. Moreover, for μ_1 large enough all the zeros of g in $[x_{\jmath_{\mu_1}}, b)$ have multiplicity one, and $\{g = 0\} \cap [x_{\jmath_{\mu_1}}, b) = \Omega \cap [x_{\jmath_{\mu_1}}, b)$.*
(b) *If g does not go to* 0 *to the left, then g has no zero interval in $(a, x_{\imath_1 - 1}]$. Moreover, for μ_2 large enough all the zeros of g in $(a, x_{\imath_{\mu_2}}]$ have multiplicity one, and $\{g = 0\} \cap (a, x_{\imath_{\mu_2}}] = \Omega \cap (a, x_{\imath_{\mu_2}}]$.*

Proof. To prove (a) assume that $[x_i, x_j]$ is a zero interval of g contained in $[x_{\jmath_1 - 1}, b)$. Then we define

$$g_0(x) := \begin{cases} g(x), & x \in (a, x_i), \\ 0, & x \in [x_i, b). \end{cases}$$

As g does not go to 0 to the right, it is clear that g and g_0 are two different extensions of G satisfying (7), in contradiction with Theorem 4.19. Thus, g has no zero interval in $[x_{\jmath_1 - 1}, b)$. Hence, the zeros of g in $(x_{\jmath_1 - 1}, b)$ are isolated, and (7) implies

$$\Omega_{[\jmath_1, j]} \leqslant Z_{[\jmath_1, j]}(g) \quad \text{for every } j > \jmath_1.$$

From Lemma 4.11(a),

$$Z^*_{[\jmath_1, j]}(g) \leqslant j - \jmath_1 + n - 2 \quad \text{for every } j > \jmath_1.$$

Then since $\{u, \Omega\}$ is a reference pair, we have

$$j - \jmath_1 - n + 1 \leqslant \Omega_{[\jmath_1, j]} \leqslant Z_{[\jmath_1, j]}(g) \leqslant Z^*_{[\jmath_1, j]}(g) \leqslant j - \jmath_1 + n - 2.$$

Therefore, for every $j > \jmath_1$,

$$Z^*_{[\jmath_1, j]}(g) - Z_{[\jmath_1, j]}(g) \leqslant 2n - 3 \quad \text{and} \quad Z_{[\jmath_1, j]}(g) - \Omega_{[\jmath_1, j]} \leqslant 2n - 3.$$

If g has a zero of multiplicity greater than one in $[x_{\jmath_\mu}, b)$ for all $\mu \in \mathbb{N}$, then

$$Z^*_{[\jmath_1, j]}(g) - Z_{[\jmath_1, j]}(g) \to +\infty \quad \text{as } j \to +\infty,$$

a contradiction. Thus, for ν large enough all the zeros of g in $[x_{\jmath_\nu}, b)$ have multiplicity one.

By (7), $\{g=0\}\cap[x_{J_1},b)\supseteq\Omega\cap[x_{J_1},b)$. Hence, if $\{g=0\}\cap[x_{J_\mu},b)\neq\Omega\cap[x_{J_\mu},b)$ for all $\mu\in\mathbb{N}$ then

$$Z_{[J_1,j]}(g)-\Omega_{[J_1,j]}\to+\infty \quad \text{as } j\to+\infty,$$

which contradicts that $Z_{[J_1,j]}(g)-\Omega_{[J_1,j]}\leqslant 2n-3$ for every $j>J_1$. Thus, for κ large enough, $\{g=0\}\cap[x_{J_\kappa},b)=\Omega\cap[x_{J_\kappa},b)$. Consequently, (a) is proved. The proof of (b) is similar. $\square$

Lemma 4.24. *Let V and G be in $\mathcal{S}_1\backslash\{0\}$. Assume that all the zeros of $v:=\Psi(V)$ are isolated and of multiplicity one. Suppose also that for some $\mu\in\mathbb{N}$ the restriction of $g:=\Psi(G)$ to (x_{I_μ},x_{J_μ}) has $m_\mu-t$ sign changes, $t\geqslant 1$.*

(a) *If g has a double zero $z'\in((x_{I_\kappa},x_{J_\kappa})\backslash\Omega)\cup(x_{I_\mu},x_{J_\mu})$ for a $\kappa>\mu$, then $g-\varepsilon v$ has at least $m_\kappa-t+2$ sign changes on $(x_{I_\kappa},x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign.*
(b) *If g has a double zero $z''\in\Omega\cap((x_{I_\kappa},x_{J_\kappa})\backslash(x_{I_\mu},x_{J_\mu}))$ for a $\kappa>\mu$, then $g-\varepsilon v$ has at least $m_\kappa-t+1$ sign changes on $(x_{I_\kappa},x_{J_\kappa})$ for all ε sufficiently small. If in addition $g=0$ on $[x_{J_\kappa},b)$ (on $(a,x_{I_\kappa}]$), then $g-\varepsilon v$ has at least $m_\kappa-t+2$ sign changes on $(x_{I_\kappa},x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign.*
(c) *If g has a simple zero $z^*\in(x_{I_\kappa},x_{J_\kappa})\backslash(x_{I_\mu},x_{J_\mu})$ for a $\kappa>\mu$ and $z^*\notin\Omega$, then $g-\varepsilon v$ has at least $m_\kappa-t+1$ sign changes on $(x_{I_\kappa},x_{J_\kappa})$ for all ε sufficiently small. If in addition $g=0$ on $[x_{J_\kappa},b)$ (on $(a,x_{I_\kappa}]$), then $g-\varepsilon v$ has at least $m_\kappa-t+2$ sign changes on $(x_{I_\kappa},x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign.*
(d) *If $g=0$ on $[x_{J_\kappa},b)$ (on $(a,x_{I_\kappa}]$) for a $\kappa>\mu$, then $g-\varepsilon v$ has at least $m_\kappa-t+1$ sign changes on $(x_{I_\kappa},x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign.*

Proof. As v does not go to 0 to the right nor to the left (because it has no zero interval), the spline $g-\lambda v$ does not go to 0 to the right nor to the left, except for at most two values of λ. Excluding these two possible values of λ, we deduce from Lemma 4.23 that $g-\lambda v$ has no zero interval in $J\backslash(x_{I_1},x_{J_1})$. Furthermore, $g-\lambda v$ may have zero intervals in (x_{I_1},x_{J_1}) only for a finite number of values of λ. As a consequence, for all ε sufficiently small $g-\varepsilon v$ has no zero interval. Hence for such all ε, each sign change of $g-\varepsilon v$ is due to a simple zero of $g-\varepsilon v$. So, taking into account that all the zeros of v are isolated and of multiplicity one, and that by (7), $v(z)=0$ if $z\in\Omega\cap(J\backslash(x_{I_1},x_{J_1}))$, for a $\kappa>\mu$ the following facts can be easily proved:

(I) The $m_\mu-t$ sign changes of g on (x_{I_μ},x_{J_μ}) produce at least $m_\mu-t$ simple zeros of $g-\varepsilon v$ on (x_{I_μ},x_{J_μ}) for all ε sufficiently small.
(II) Each simple zero z of g in $(x_{I_\kappa},x_{J_\kappa})\backslash(x_{I_\mu},x_{J_\mu})$ produces, for all ε sufficiently small, at least one simple zero of $g-\varepsilon v$, say $z(\varepsilon)$, with $z(\varepsilon)\to z$ as $\varepsilon\to 0$.

(III) Each double zero z' of g in $((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash\Omega)\cup(x_{\imath_\mu}, x_{\jmath_\mu})$ produces, for all ε sufficiently small (and with a suitable sign if $v(z')\neq 0$), two simple zeros of $g-\varepsilon v$, say $z_1'(\varepsilon)$ and $z_2'(\varepsilon)$, such that $z_1'(\varepsilon)\to z'$ and $z_2'(\varepsilon)\to z'$ as $\varepsilon\to 0$.
(IV) Each double zero z'' of g in $\Omega\cap((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash(x_{\imath_\mu}, x_{\jmath_\mu}))$ produces, for all ε sufficiently small, two simple zeros of $g-\varepsilon v$, say $z_1''(\varepsilon)=z''$ and $z_2''(\varepsilon)$, such that $z_2''(\varepsilon)\to z''$ as $\varepsilon\to 0$.
(V) Each nonisolated zero of g in $\Omega\cap((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash(x_{\imath_\mu}, x_{\jmath_\mu}))$ is a simple zero of $g-\varepsilon v$ for every $\varepsilon\neq 0$.
(VI) For all ε sufficiently small (and possibly with a suitable sign if III) applies), all the simple zeros of $g-\varepsilon v$ obtained in (I)–(V) are different.

Applying (5), for every $\kappa>\mu$ we obtain

$$\Omega_{(\imath_\kappa,\imath_\mu]} = \imath_\mu - \imath_\kappa - \sigma(\imath_\kappa) + \sigma(\imath_\mu) \quad\text{and}\quad \Omega_{[\jmath_\mu,\jmath_\kappa)} = \jmath_\kappa - \jmath_\mu - \sigma(\jmath_\kappa) + \sigma(\jmath_\mu).$$

Therefore,

$$\Omega_{(\imath_\kappa,\imath_\mu]} + \Omega_{[\jmath_\mu,\jmath_\kappa)} = m_\kappa - m_\mu. \tag{9}$$

Suppose that z' is a double zero of g and $z'\in((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash\Omega)\cup(x_{\imath_\mu}, x_{\jmath_\mu})$. From (7), $g=0$ in $\Omega\cap((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash(x_{\imath_\mu}, x_{\jmath_\mu}))$. Then using (9) and (I)–(VI) we deduce that $g-\varepsilon v$ has at least $m_\kappa - t + 2$ sign changes on $(x_{\imath_\kappa}, x_{\jmath_\kappa})$ for all ε sufficiently small and with a suitable sign. This proves (a).

Assume now that $z''\in\Omega\cap((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash(x_{\imath_\mu}, x_{\jmath_\mu}))$ is a double zero of g. Then taking into account again that $g=0$ in $\Omega\cap((x_{\imath_\kappa}, x_{\jmath_\kappa})\backslash(x_{\imath_\mu}, x_{\jmath_\mu}))$, and using (9), (I), (II), (IV)–(VI) we see that $g-\varepsilon v$ has at least $m_\kappa - t + 1$ sign changes on $(x_{\imath_\kappa}, x_{\jmath_\kappa})$ for all ε sufficiently small. Suppose that in addition $g=0$ on $[x_{\jmath_\kappa}, b)$. Then there exists $j\leqslant \jmath_\kappa$ such that $g=0$ on $[x_j, b)$ and $g(x_{j-1})\neq 0$. Therefore, apart from the simple zeros of $g-\varepsilon v$ mentioned in (I), (II), (IV) and (V), $g-\varepsilon v$ has another simple zero in (x_{j-1}, x_j) for all ε sufficiently small and with a suitable sign. An analogous argument applies when $g=0$ on $(a, x_{\imath_\kappa}]$. This proves (b). The proofs of (c) and (d) are similar. □

4.2. *Levels in $\Psi(\mathscr{S}_1)$. The WT-spaces $\mathscr{S}_1^{\rho,\tau}$ and $\mathscr{T}_\ell^\tau$*

Recall that we are assuming that $\{u,\Omega\}$ is a reference pair, where $\Omega=\{z_l\}_{l\in\mathbb{Z}}$, with $a<z_i<z_j<b$ whenever $i<j$, and $z_i\to a$ as $i\to-\infty$, $z_j\to b$ as $j\to+\infty$.

Remark 4.25. Let $G\in\mathscr{S}_1$. Because of Lemma 4.23(a), if $g:=\Psi(G)$ does not go to 0 to the right then there exists $j'\in\mathbb{Z}$ such that all the zeros of g in $[z_{j'}, b)$ are isolated and of multiplicity one, and in addition $\{g=0\}\cap[z_{j'}, b)=\Omega\cap[z_{j'}, b)$. For all $l\in\mathbb{Z}$, $\bar{z}_l$ will henceforth denote an arbitrary point in (z_l, z_{l+1}). Accordingly, if g does not go to 0 to the right then there exists $j' \in \mathbb{Z}$ such that $g(\bar{z}_j)g(\bar{z}_{j+1})<0$ for every $j \geqslant j'$. Analogously, using now Lemma 4.23(b) we see that if g does not go to 0 to the left then there exists $i' \in \mathbb{Z}$ such that $g(\bar{z}_i)g(\bar{z}_{i-1})<0$ for every $i \leqslant i'$.

The following theorem will play a decisive role in the proof of Theorem 4.12.

Theorem 4.26. *Let* $g_1 := \Psi(G_1)$, $g_2 := \Psi(G_2)$, *where* $G_1, G_2 \in \mathscr{S}_1$.

(a) *Assume* g_2 *does not go to* 0 *to the right. Then there exists*

$$l_{\text{right}} := \lim_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)}, \quad -\infty \leqslant l_{\text{right}} \leqslant +\infty.$$

(b) *Assume* g_2 *does not go to* 0 *to the left. Then there exists*

$$l_{\text{left}} := \lim_{\substack{x\downarrow a\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)}, \quad -\infty \leqslant l_{\text{left}} \leqslant +\infty.$$

Proof. To prove (a) assume that $g_2 - \theta g_1$ does not go to 0 to the right for any $\theta \in \mathbb{R}\backslash\{0\}$. Otherwise $l_{\text{right}} = 1/\theta$ for some θ, and therefore (a) is obvious. Then since both g_2 and $g_2 - \theta g_1$ do not go to 0 to the right, using Remark 4.25 we deduce that there exists $j'(\theta) \in \mathbb{Z}$ such that for every $j \geqslant j'(\theta)$,

$$g_2(\bar{z}_j)g_2(\bar{z}_{j+1})<0 \quad \text{and} \quad [g_2(\bar{z}_j) - \theta g_1(\bar{z}_j)][g_2(\bar{z}_{j+1}) - \theta g_1(\bar{z}_{j+1})]<0.$$

Hence, either

$$0<\frac{g_2(x) - \theta g_1(x)}{g_2(x)} = 1 - \frac{\theta g_1(x)}{g_2(x)} \quad \text{for every } x \in (z_{j'(\theta)}, b)\backslash\Omega$$

or

$$0>\frac{g_2(x) - \theta g_1(x)}{g_2(x)} = 1 - \frac{\theta g_1(x)}{g_2(x)} \quad \text{for every } x \in (z_{j'(\theta)}, b)\backslash\Omega.$$

Suppose

$$\liminf_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)}<\gamma<\limsup_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)}, \quad \gamma\neq 0. \tag{10}$$

We have just proven that there exists $j'(1/\gamma) \in \mathbb{Z}$ such that either

$$\frac{g_1(x)}{\gamma g_2(x)}<1 \quad \text{for every } x \in (z_{j'(1/\gamma)}, b)\backslash\Omega$$

or

$$\frac{g_1(x)}{\gamma g_2(x)}>1 \quad \text{for every } x \in (z_{j'(1/\gamma)}, b)\backslash\Omega.$$

In either case (10) is contradicted. Thus,

$$\liminf_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)} = \limsup_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)},$$

and hence the limit exists. So (a) is proved. The proof of (b) is similar. □

Definition 4.27. Let $G_1, G_2 \in \mathscr{S}_1$. We say that $g_1 := \Psi(G_1)$ and $g_2 := \Psi(G_2)$ have the *same right level* (*same left level*) if either g_1 and g_2 go to 0 to the right (to the left) or g_2 does not go to 0 to the right (to the left) and

$$0 < \lim_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{|g_1(x)|}{|g_2(x)|} < \infty \quad \left(0 < \lim_{\substack{x\uparrow a\\ x\notin\Omega}} \frac{|g_1(x)|}{|g_2(x)|} < \infty\right).$$

The functions g_1 and g_2 have the *same level* if they have the same right and left levels. Finally, we say that the function g_1 has a *lower right level than* g_2 (*lower left level than* g_2) if g_2 does not go to 0 to the right (to the left) and

$$\lim_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)} = 0 \quad \left(\lim_{\substack{x\uparrow b\\ x\notin\Omega}} \frac{g_1(x)}{g_2(x)} = 0\right).$$

Lemma 4.28. *Let $G_1, G_2 \in \mathscr{S}_1$, and let $g_1 := \Psi(G_1)$, $g_2 := \Psi(G_2)$.*

(a) *Assume that for some $\mu \in \mathbb{N}$ and for all ε sufficiently small,*[2] *g_1, g_2, and $g_1 - \varepsilon g_2$ have no double zero in $[x_{J_\mu}, b)$, and*

$$\begin{aligned}\{g_1 = 0\} \cap [x_{J_\mu}, b) &= \{g_2 = 0\} \cap [x_{J_\mu}, b)\\ &= \{g_1 - \varepsilon g_2 = 0\} \cap [x_{J_\mu}, b) = \Omega \cap [x_{J_\mu}, b).\end{aligned}$$

Then g_1 does not have a lower right level than g_2.

(b) *Assume that for some $\mu \in \mathbb{N}$ and for all ε sufficiently small, g_1, g_2 and $g_1 - \varepsilon_2 g_2$ have no double zero in $(a, x_{\imath_\mu}]$, and*

$$\begin{aligned}\{g_1 = 0\} \cap (a, x_{\imath_\mu}] &= \{g_2 = 0\} \cap (a, x_{\imath_\mu}]\\ &= \{g_1 - \varepsilon_2 g_2 = 0\} \cap (a, x_{\imath_\mu}] = \Omega \cap (a, x_{\imath_\mu}].\end{aligned}$$

Then g_1 does not have a lower left level than g_2.

[2] This means $\varepsilon \neq 0$, $|\varepsilon|$ small enough.

(c) *Assume that for some* $\mu \in \mathbb{N}$ *and for all* ε *sufficiently small*, g_1, g_2 *and* $g_1 - \varepsilon g_2$ *have no double zero in* $J \backslash (x_{\iota_\mu}, x_{J_\mu})$, *and*

$$\begin{aligned}\{g_1 = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) &= \{g_2 = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) \\ &= \{g_1 - \varepsilon g_2 = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) \\ &= \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})).\end{aligned}$$

Then g_1 *does not have a lower right or left level than* g_2.

Proof. Recall that $\Omega = \{z_j\}_{j \in \mathbb{Z}}$ and that for all $j \in \mathbb{Z}$, $\bar{z}_j$ denotes an arbitrary point in (z_j, z_{j+1}). To prove (a), consider a $z_{j'} \in [x_{J_\mu}, b)$. Then we see that for all ε sufficiently small, g_1, g_2, and $g_1 - \varepsilon g_2$ have no double zero in $[z_{j'}, b)$, and

$$\{g_1 = 0\} \cap [z_{j'}, b) = \{g_2 = 0\} \cap [z_{j'}, b) = \{g_1 - \varepsilon g_2 = 0\} \cap [z_{j'}, b) = \Omega \cap [z_{j'}, b).$$

Therefore, for all $j \geqslant j'$ and for all ε sufficiently small we get

(i) $g_1(\bar{z}_j) g_1(\bar{z}_{j+1}) < 0$,
(ii) $g_2(\bar{z}_j) g_2(\bar{z}_{j+1}) < 0$, and
(iii) $[g_1(\bar{z}_j) - \varepsilon g_2(\bar{z}_j)][g_1(\bar{z}_{j+1}) - \varepsilon g_2(\bar{z}_{j+1})] < 0$.

Take now an ε_1 so that (iii) is valid with $\varepsilon = \varepsilon_1$, and in addition

$$\varepsilon_1 g_1 g_2 > 0 \quad \text{and} \quad |g_1| > |\varepsilon_1 g_2| \qquad \text{on } (z_{j'}, z_{j'+1}). \tag{11}$$

Then using (i), (ii) and the first inequality in (11), we deduce that

$$\varepsilon_1 g_1 g_2 > 0 \qquad \text{on } (z_{j'}, b) \backslash \Omega.$$

From (11) it is easy to check that $g_1(g_1 - \varepsilon_1 g_2) > 0$ on $(z_{j'}, z_{j'+1})$. Hence, using (i), and (iii) with $\varepsilon = \varepsilon_1$, we obtain

$$g_1(g_1 - \varepsilon_1 g_2) > 0 \quad \text{on } (z_{j'}, b) \backslash \Omega.$$

Finally, as

$$\varepsilon_1 g_1 g_2 > 0 \qquad \text{and } g_1(g_1 - \varepsilon_1 g_2) > 0 \quad \text{on } (z_{j'}, b) \backslash \Omega,$$

it is easy to see that $|g_1| > |\varepsilon_1 g_2|$ on $(z_{j'}, b) \backslash \Omega$, whence

$$\lim_{\substack{x \uparrow b \\ x \notin \Omega}} \frac{|g_1(x)|}{|g_2(x)|} > 0.$$

Thus, g_1 does not have a lower right level than g_2. This proves (a). The proof of (b) is similar.

To prove (c), note that the hypotheses of (a) and (b) hold simultaneously, and thus g_1 does not have a lower right or left level than g_2. $\square$

Remark 4.29. We now construct a suitable basis for $\mathscr{S}_1$. We choose $m_1 - 1$ points $\{y_k\}_{k=1}^{m_1-1}$ in the interval $(x_{\imath_1^+}, x_{\jmath_1^-})$ in the following way. Take n points in (x_{-1}, x_1) and one point in each of the *components* (x_i, x_{i+1}), $i = \imath_1^+, \imath_1^+ + 1, \dots, -3, -2$ and $i = 1, 2, \dots, \jmath_1^- - 2, \jmath_1^- - 1$ in such a manner that $y_1 < y_2 < \cdots < y_{m_1-1}$. We shall use these points, together with a set of *replacement* points $y'_1 < y'_2 < \dots < y'_{m_1-1}$ to obtain the splines in the basis for $\mathscr{S}_1$, where $\{y'_k\}_{k=1}^{m_1-1} \cap \{y_k\}_{k=1}^{m_1-1} = \emptyset$ and each y'_k is taken in the same component in which y_k is. Using Property [JKZ] in the m_1-dimensional WT-space $\mathscr{S}_1$ we obtain a spline $V_0 \in \mathscr{S}_1 \backslash \{0\}$ for which $\{y_k\}_{k=1}^{m_1-1}$ is an alternating set. Note that by the location of the points $\{y_k\}_{k=1}^{m_1-1}$, if V_0 has a zero interval then V_0 vanishes identically. Thus, V_0 has no zero interval. Applying now Lemma 4.11(a) to the restriction of V_0 to $(x_{\imath_1^+}, x_{\jmath_1^-})$, which is in $\mathscr{S}_{\imath_1^+, \jmath_1^-}$, we see that $\{y_k\}_{k=1}^{m_1-1}$ are the only zeros (of multiplicity 1) of V_0 in $[x_{\imath_1^+}, x_{\jmath_1^-}]$. We now extend $\{V_0\}$ to a basis for $\mathscr{S}_1$. For each $l = 1, 2, \dots, m_1 - 1$, consider the set $\{y_k^l\}_{k=1}^{m_1-1}$, with $y_k^l = y_k$ for $k \neq l$ and $y_l^l = y'_l$. From this set, V_l is obtained in the same way as V_0, and therefore it has the same properties; V_l has the only zeros (of multiplicity 1) $y_1^l, y_2^l, \dots, y_{m_1-1}^l$ in $[x_{\imath_1^+}, x_{\jmath_1^-}]$. It is thus easy to see that the set $\{V_0, V_1, \dots, V_{m_1-1}\}$ is linearly independent and it is therefore a basis for $\mathscr{S}_1$. We say that this basis is obtained by the *replacement method* based on the set $\{y_k\}_{k=1}^{m_1-1}$, with the replacement points $y'_1, y'_2, \dots, y'_{m_1-1}$. For $l = 0, 1, \dots, m_1 - 1$, every V_l has $m_1 - 1$ sign changes and has no zero interval. Let $v_l := \Psi(V_l)$, $l = 0, 1, \dots, m_1 - 1$. Then $\{v_0, v_1, \dots, v_{m_1-1}\}$ is a basis for $\Psi(\mathscr{S}_1)$. In the following theorem we shall use the splines $\{v_l\}_{l=0}^{m_1-1}$, as well as the points $\{y_k^l\}_{k=1}^{m-1}$ for each $l = 1, 2, \dots, m_1 - 1$.

Theorems 4.30 and 4.36 are the key to constructing appropriate WT-subspaces of $\mathscr{S}_1$. In one of these subspaces we will find an H_0 with the property that $h_0 := \Psi(H_0)$ makes Theorem 4.12 to hold. Theorem 4.30 establishes five properties for, in particular, the space $\mathscr{S}_1$ $(= \mathscr{S}_1^{0,0})$. We will prove in Lemma 4.33 that $\mathscr{S}_1$ indeed satisfies these properties.

Theorem 4.30. *Assume that* $\mathscr{S}_1^{0,\tau}$ *is an* $(m_1 - \tau)$*-dimensional WT-subspace of* $\mathscr{S}_1$, $0 \leqslant \tau < n - 1 - \sigma(\jmath_1)$, *with a basis* $\{V_0^{0,\tau}, V_1^{0,\tau}, \dots, V_{m_1-\tau-1}^{0,\tau}\}$, *and such that the following properties hold*:

(a$_1$) *For each* $l = 0, 1, \dots, m_1 - \tau - 1$, *the spline* $V_l^{0,\tau}$ *changes sign at the* $m_1 - \tau - 1$ *points* $y_1^l, y_2^l, \dots, y_{m_1-\tau-1}^l$, *whence it has* $m_1 - \tau - 1$ *sign changes. Moreover, for each* $l = 0, 1, \dots, m_1 - \tau - 1$, $v_l^{0,\tau} := \Psi(V_l^{0,\tau})$ *is proportional to* v_l *on* $(a, x_{\jmath_1^- - \tau}]$, *and* $v_l^{0,\tau}$ *has no zero interval in* $(a, x_{\jmath_1}]$.

(a$_2$) *Let* $G \in \mathscr{S}_1^{0,\tau}$. *Then for every* $v \in \mathbb{N}$, *the restriction of* $g := \Psi(G)$ *to* $(x_{\imath_v}, x_{\jmath_v})$ *has at most* $m_v - \tau - 1$ *sign changes.*

(a$_3$) *Let* $G \in \mathscr{S}_1^{0,\tau}$ *and assume that for some* $\mu \in \mathbb{N}$, *the restriction of* $g\ (= \Psi(G))$ *to* $(x_{\iota_\mu}, x_{J_\mu})$ *has* $m_\mu - \tau - 2$ *sign changes. If* g *does not go to* 0 *to the right nor to the left and the broken line* $g^{(n-2)}$ *has no zero in some interval* $[x_{i_1}, x_{i_1+1}]$, *then* g *has no double zero, and*

$$\{g = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})).$$

(a$_4$) *Let* $G \in \mathscr{S}_1^{0,\tau}$ *and assume that for some* $\mu \in \mathbb{N}$, *the restriction of* $g\ (= \Psi(G))$ *to* $(x_{\iota_\mu}, x_{J_\mu})$ *has* $m_\mu - \tau - 1$ *sign changes. Suppose also that* g *does not go to* 0 *to the right nor to the left. Then* g *has no double zero, and*

$$\{g = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})).$$

Moreover, $g^{(n-2)}$ *has at least a zero in* $[x_i, x_{i+1}]$, *all* $i \in \mathbb{Z}$.

(a$_5$) *For each* $l = 0, 1, \ldots, m_1 - \tau - 1$, *if* $v_l^{0,\tau}\ (= \Psi(V_l^{0,\tau}))$ *does not go to* 0 *to the right, then* $v_l^{0,\tau}$ *has no zero interval, all the zeros of* $v_l^{0,\tau}$ *have multiplicity one, and*

$$\{v_l^{0,\tau} = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

If $v_0^{0,\tau}$ *does not go to* 0 *to the right, then*

(i) *The set* $\mathscr{S}_1^{0,\tau+1} := \{G \in \mathscr{S}_1^{0,\tau} : \Psi(G)$ *has a lower right level than* $v_0^{0,\tau}\}$ *is an* $(m_1 - \tau - 1)$*-dimensional WT-subspace of* $\mathscr{S}_1^{0,\tau}$.

(ii) *There exists a basis* $\{V_0^{0,\tau+1}, V_1^{0,\tau+1}, \ldots, V_{m_1-\tau-2}^{0,\tau+1}\}$ *for* $\mathscr{S}_1^{0,\tau+1}$ *such that* $\mathscr{S}_1^{0,\tau+1}$ *and this basis fulfill the analogs of* (a$_1$)–(a$_5$) *with* $\tau + 1$ *in place of* τ, *say* (a$'_1$)–(a$'_5$), *respectively.*

Before proving Theorem 4.30, we need to show some results.

Proposition 4.31. *Let* $\mathscr{S}_1^{0,\tau}$ *be an* $(m_1 - \tau)$*-dimensional WT-subspace of* $\mathscr{S}_1$, $0 \leqslant \tau \leqslant n - 1 - \sigma(J_1)$, *with a basis* $\{V_0^{0,\tau}, V_1^{0,\tau}, \ldots, V_{m_1-\tau-1}^{0,\tau}\}$. *Assume that* $\mathscr{S}_1^{0,\tau}$ *and this basis satisfy* (a$_1$) *of Theorem* 4.30, *and let* $G_0 \in \mathscr{S}_1^{0,\tau}$. *If* $[x_{\iota_1}, x_{j'}]$ *is a zero interval of* G_0, $J_1^- - \tau \leqslant j' \leqslant J_1$, *then* $\Psi(G_0)$ *vanishes identically on* $[x_{\iota_1}, b)$.

Proof. Let $G_0 \in \mathscr{S}_1^{0,\tau}$ and assume that $[x_{\iota_1}, x_{j'}]$ is a zero interval of G_0, $J_1^- - \tau \leqslant j' \leqslant J_1$. We claim that $G_0 = 0$ necessarily. Suppose, contrary to our claim, that $[x_{\iota_1}, x_{i_0}]$ is a zero interval of G_0, with $J_1^- - \tau \leqslant i_0 < J_1$ and $G_0(x_{i_0+1}) \neq 0$. On the other hand, it follows from (a$_1$) that $V_0^{0,\tau}$ has simple zeros at the points $y_1, y_2, \ldots, y_{m_1-\tau-1}$, which are in $(x_{\iota_1}, x_{J_1^- - \tau}) \subseteq (x_{\iota_1}, x_{i_0})$. Then $G_0 - \varepsilon V_0^{0,\tau}$ is in $\mathscr{S}_1^{0,\tau}$ and it has at least $m_1 - \tau - 1$ sign changes on (x_{ι_1}, x_{i_0}) for all $\varepsilon \neq 0$. Moreover, since $G_0(x_{i_0+1}) \neq 0$, it is easy to see that we can choose ε' sufficiently small and with a suitable sign in such a way that $G_0 - \varepsilon' V_0^{0,\tau}$ has another sign change on (x_{i_0}, x_{i_0+1}). Therefore, $G_0 - \varepsilon' V_0^{0,\tau}$ has at least

$m_1 - \tau$ sign changes, which contradicts that $G_0 - \varepsilon' V_0^{0,\tau}$ is in the $(m_1 - \tau)$-dimensional WT-space $\mathscr{S}_1^{0,\tau}$. Thus, $G_0 = 0$ and so the claim is proved. Then it follows from Theorem 4.19 that $\Psi(G_0)$ vanishes identically on $[x_{\iota_1}, b)$. $\square$

Lemma 4.32. *Let $\mathscr{S}_1^{0,\tau}$ be an $(m_1 - \tau)$-dimensional WT-subspace of $\mathscr{S}_1$, $0 \leqslant \tau \leqslant n - 1 - \sigma(J_1)$, with a basis $\{V_0^{0,\tau}, V_1^{0,\tau}, \dots, V_{m_1-\tau-1}^{0,\tau}\}$. If $\mathscr{S}_1^{0,\tau}$ and this basis satisfy* (a$_1$)–(a$_5$) *of Theorem* 4.30, *then there holds*

(a) *The splines $v_l^{0,\tau}$ $(= \Psi(V_l^{0,\tau})$, $l = 0, 1, \dots, m_1 - \tau - 1$, have the same right level. Hence, $v_0^{0,\tau}$ does not have a lower right level than $\Psi(G)$ for any $G \in \mathscr{S}_1^{0,\tau}$.*
(b) *Let $G_1 \in \mathscr{S}_1^{0,\tau}$ and suppose that $g_1 := \Psi(G_1)$ has $m_\mu - \tau - 1$ sign changes on $(x_{\iota_\mu}, x_{J_\mu})$ for some $\mu \in \mathbb{N}$. Then g_1 has the same right level as $v_0^{0,\tau}$.*
(c) *Let $G_0 \in \mathscr{S}_1^{0,\tau} \backslash \{0\}$, and suppose that $[x_{\iota_1}, x_{J_1^- - \tau - 1}]$ is a zero interval of G_0. Then $\Psi(G_0)$ has the same right level as $v_0^{0,\tau}$.*

To prove (a) and (b) we shall use the following claim.

Claim 1. *Let $G_1 \in \mathscr{S}_1^{0,\tau}$ with $m_\mu - \tau - 1$ sign changes on $(x_{\iota_\mu}, x_{J_\mu})$ for some $\mu \in \mathbb{N}$. If $v_{l'}^{0,\tau}$ $(= \Psi(V_{l'}^{0,\tau}))$ does not go to 0 to the right for some l', $0 \leqslant l' \leqslant m_1 - \tau - 1$, then $g_1 := \Psi(G_1)$ does not go to 0 to the right nor to the left.*

To prove the claim, suppose $v_{l'}^{0,\tau}$ does not go to 0 to the right. So taking into account (a$_5$) we can use Lemma 4.24 with $G = G_1$, $V = V_{l'}^{0,\tau}$ and $t = \tau + 1$. Assume now that g_1 goes to 0 to the right. Then using (d) in Lemma 4.24 we deduce that there exists a $\kappa > \mu$ and large enough for which $g_1 - \varepsilon v_{l'}^{0,\tau}$ has at least $m_\kappa - \tau$ sign changes on $(x_{\iota_\kappa}, x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign. But this is in contradiction with (a$_2$) applied to $G_1 - \varepsilon V_{l'}^{0,\tau} \in \mathscr{S}_1^{0,\tau}$. The same argument proceeds if g_1 goes to 0 to the left. Therefore, the claim is proved.

Observe that, due to (a$_1$), for $l = 0, 1, \dots, m_1 - \tau - 1$ the spline $V_l^{0,\tau}$ satisfies the hypothesis on G_1 in Claim 1. Hence any result we get for G_1 (g_1) is also valid for each $V_l^{0,\tau}$ ($v_l^{0,\tau}$). For instance, we deduce that $v_l^{0,\tau}$ goes to 0 to the right if and only if $v_0^{0,\tau}$ goes to 0 to the right. If $v_l^{0,\tau}$ goes to 0 to the right for every $l = 0, 1, \dots, m_1 - \tau - 1$, then it is clear that any G in $\mathscr{S}_1^{0,\tau}$ satisfies that $\Psi(G)$ also goes to 0 to the right, and therefore $\Psi(G)$ has the same right level as $v_0^{0,\tau}$. Thus (a) and (b) are true in this case. Assume now that $v_l^{0,\tau}$ does not go to 0 to the right for any $l = 0, 1, \dots, m_1 - \tau - 1$. Let G_1 be in $\mathscr{S}_1^{0,\tau}$ with $m_\mu - \tau - 1$ sign changes on $(x_{\iota_\mu}, x_{J_\mu})$ for some $\mu \in \mathbb{N}$. Then Claim 1 implies that g_1 $(= \Psi(G_1))$ does not go to 0 to the right nor to the left. Accordingly, for every $l = 0, 1, \dots, m_1 - \tau - 1$, each of the splines G_1 and $V_l^{0,\tau}$

satisfies the hypotheses of (a_4), and it is not difficult to see that also $G_1 - \varepsilon V_l^{0,\tau}$ satisfies the hypotheses of (a_4) for all ε sufficiently small. Hence, we deduce that g_1, $v_l^{0,\tau}$, and $g_1 - \varepsilon v_l^{0,\tau}$ for all ε sufficiently small, have no double zeros and

$$\{g_1 = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) = \{v_l^{0,\tau} = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) = \{g_1 - \varepsilon v_l^{0,\tau} = 0\} \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu})).$$

Therefore, Lemma 4.28(c) shows the following assertion: g_1 does not have a lower right (or left) level than $v_l^{0,\tau}$, $l = 0, 1, \ldots, m_1 - \tau - 1$. In particular it follows that for $l = 0, 1, \ldots, m_1 - \tau - 1$, the splines $v_l^{0,\tau}$ have the same right (and left) level. Hence, it is easy to see that $v_0^{0,\tau}$ does not have a lower right (or left) level than $\Psi(G)$ for any $G \in \mathcal{S}_1^{0,\tau}$. This proves (a). Now both (a) and the above assertion prove (b).

To prove (c), let $G_0 \in \mathcal{S}_1^{0,\tau} \backslash \{0\}$, and suppose that $[x_{\imath_1}, x_{J_1}^- - \tau - 1]$ is a zero interval of G_0. Note that from (a) it is sufficient to show that $g_0 := \Psi(G_0)$ does not have a lower right level than $v_0^{0,\tau}$. This is immediate if $v_0^{0,\tau}$ goes to 0 to the right. So assume that $v_0^{0,\tau}$ does not go to 0 to the right. Under this assumption, we first demonstrate the following two claims.

Claim 2. *For all ε sufficiently small, $g_0 - \varepsilon v_0^{0,\tau}$ has no double zero, and*

$$\{g_0 - \varepsilon v_0^{0,\tau} = 0\} \cap (J \backslash (x_{\imath_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\imath_1}, x_{J_1})).$$

To prove the claim, note first that $g_0 - \varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left for all ε sufficiently small because $v_0^{0,\tau}$ does not go to 0 to the right nor to the left. From (a_1) and the location of the points $y_1, y_2, \ldots, y_{m_1-\tau-1}$, $V_0^{0,\tau}$ has $m_1 - \tau - 2$ sign changes on $(x_{\imath_1^+}, x_{J_1^- - \tau - 1})$. Then for all $\varepsilon \neq 0$, the spline $G_0 - \varepsilon V_0^{0,\tau}$ has $m_1 - \tau - 2$ sign changes on $(x_{\imath_1^+}, x_{J_1^- - \tau - 1})$ since $[x_{\imath_1}, x_{J_1^- - \tau - 1}]$ is a zero interval of G_0. Furthermore, as $G_0 \neq 0$, from Proposition 4.31 we deduce that $[x_{\imath_1}, x_{J_1^- - \tau - 1}]$ is a maximal zero interval of G_0. Therefore, it is clear that for all $\varepsilon \neq 0$, small enough and with a suitable sign, $G_0 - \varepsilon V_0^{0,\tau}$ has a sign change on $(x_{J_1^- - \tau - 1}, x_{J_1^- - \tau})$. Thus, there exists an ε_0 sufficiently small for which $g_0 - \varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left and $G_0 - \varepsilon V_0^{0,\tau}$ has $m_1 - \tau - 1$ sign changes on $(x_{\imath_1^+}, x_{J_1^- - \tau})$ for all ε satisfying $0 < \varepsilon\varepsilon_0 \leqslant \varepsilon_0^2$. Then applying ($a_2$) to $G_0 - \varepsilon V_0^{0,\tau}$, in $\mathcal{S}_1^{0,\tau}$, we see that $G_0 - \varepsilon V_0^{0,\tau}$ has $m_1 - \tau - 1$ sign changes on $(x_{\imath_1}, x_{J_1})$. Accordingly, for all ε satisfying $0 < \varepsilon\varepsilon_0 \leqslant \varepsilon_0^2$, using ($a_4$) we deduce that $g_0 - \varepsilon v_0^{0,\tau}$ has no double zero, and

$$\{g_0 - \varepsilon v_0^{0,\tau} = 0\} \cap (J \backslash (x_{\imath_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\imath_1}, x_{J_1})).$$

Now, in order to complete the proof of the claim, we show that there exists an ε_1 sufficiently small and satisfying $\varepsilon_0\varepsilon_1 < 0$ for which $g_0 - \varepsilon v_0^{0,\tau}$ has no double zero, and

$$\{g_0 - \varepsilon v_0^{0,\tau} = 0\} \cap (J \backslash (x_{\imath_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\imath_1}, x_{J_1}))$$

for all ε satisfying $0<\varepsilon\varepsilon_1\leqslant\varepsilon_1^2$. As $[x_{\imath_1}, x_{J_1^- -\tau-1}]$ is a maximal zero interval of G_0, the broken line $G_0^{(n-2)}$ vanishes identically on $[x_{\imath_1}, x_{J_1^- -\tau-1}]$ and it has no zero in $(x_{J_1^- -\tau-1}, x_{J_1^- -\tau}]$. Suppose, without loss of generality, $G_0^{(n-2)}>0$ on $(x_{J_1^- -\tau-1}, x_{J_1^- -\tau}]$. For all ε satisfying $0<\varepsilon\varepsilon_0\leqslant\varepsilon_0^2$, the spline $G_0-\varepsilon V_0^{0,\tau}$ has $m_1-\tau-1$ sign changes on $(x_{\imath_1^+}, x_{J_1^- -\tau})$. Then applying $n-2$ times Lemma 4.8 we see that the broken line $(G_0-\varepsilon V_0^{0,\tau})^{(n-2)}$ has a simple zero in each (x_i, x_{i+1}), $i=\imath_1^+, \imath_1^+ +1, \dots, J_1^- -\tau-1$. Hence, $(G_0-\varepsilon' V_0^{0,\tau})^{(n-2)}(x_{J_1^- -\tau-1})<0$ for all ε' sufficiently small and satisfying $0<\varepsilon'\varepsilon_0\leqslant\varepsilon_0^2$, because $(G_0-\varepsilon' V_0^{0,\tau})^{(n-2)}(x_{J_1^- -\tau})>0$. Then it is not difficult to see that there exists a sufficiently small ε_1 satisfying $\varepsilon_1\varepsilon_0<0$ for which $g_0-\varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left and $(G_0-\varepsilon V_0^{0,\tau})^{(n-2)}>0$ on $[x_{J_1^- -\tau-1}, x_{J_1^- -\tau}]$ for all ε satisfying $0<\varepsilon\varepsilon_1\leqslant\varepsilon_1^2$. Hence, using (a$_4$) we deduce that $G_0-\varepsilon V_0^{0,\tau}$ has $m_1-\tau-2$ sign changes on $(x_{\imath_1}, x_{J_1})$ for all ε satisfying $0<\varepsilon\varepsilon_1\leqslant\varepsilon_1^2$, because $G_0-\varepsilon V_0^{0,\tau}$ has $m_1-\tau-2$ sign changes on $(x_{\imath_1}, x_{J_1^- -\tau-1})$ for all $\varepsilon\neq 0$. Accordingly, for all ε satisfying $0<\varepsilon\varepsilon_1\leqslant\varepsilon_1^2$, (a$_3$) applied to $G_0-\varepsilon V_0^{0,\tau}$ shows that $g_0-\varepsilon v_0^{0,\tau}$ has no double zero, and $\{g_0-\varepsilon v_0^{0,\tau}=0\}\cap(J\backslash(x_{\imath_1}, x_{J_1}))=\Omega\cap(J\backslash(x_{\imath_1}, x_{J_1}))$. So the claim is proved.

Claim 3. *The spline g_0 has no zero interval in $[x_{J_1^- -\tau-1}, b)$.*

Assume, contrary to our claim, that g_0 has a zero interval in $[x_{J_1^- -\tau-1}, b)$. By hypothesis, $[x_{\imath_1}, x_{J_1^- -\tau-1}]$ is a zero interval of $G_0\in\mathscr{S}_1^{0,\tau}\backslash\{0\}$. Therefore, from Proposition 4.31, $[x_{\imath_1}, x_{J_1^- -\tau-1}]$ is a maximal zero interval of G_0. Then, according to our assumption, there exists a $j_0>J_1^- -\tau-1$ such that g_0 has no zero interval in $[x_{J_1^- -\tau-1}, x_{j_0}]$ and the restriction of g_0 to $[x_{J_1^- -\tau-1}, x_{j_0}]$ is in $\mathscr{S}^0_{J_1^- -\tau-1, j_0}$. We first show that $j_0\leqslant J_1$ is not possible. Indeed, suppose $j_0\leqslant J_1$. Recall that $G_0-\varepsilon V_0^{0,\tau}$ has $m_1-\tau-2$ sign changes on $(x_{\imath_1}, x_{J_1^- -\tau-1})$ for all $\varepsilon\neq 0$. As G_0 is in $\mathscr{S}^0_{J_1^- -\tau-1, j_0}$, it follows that for all ε sufficiently small and with a suitable sign, $G_0-\varepsilon V_0^{0,\tau}$ has another sign change on $(x_{J_1^- -\tau-1}, x_{J_1^- -\tau})$. The same fact is valid on (x_{j_0-1}, x_{j_0}), but both signs of ε cannot be the same, because $G_0-\lambda V_0^{0,\tau}$, in $\mathscr{S}_1^{0,\tau}$, cannot have $m_1-\tau$ sign changes for any λ. We conclude that $G_0-\varepsilon V_0^{0,\tau}$ has $m_1-\tau-1$ sign changes for all ε sufficiently small. This contradicts the last sentence of (a$_4$) because we have seen in the proof of Claim 2 that for all ε sufficiently small and with a suitable sign, $(G_0-\varepsilon V_0^{0,\tau})^{(n-2)}$ has no zero in $[x_{J_1^- -\tau-1}, x_{J_1^- -\tau}]$. Thus $j_0>J_1$. In this case, we will also obtain a contradiction. Indeed, if $j_0>J_1$, then it is easy to see that for all $\varepsilon\neq 0$, small enough and with a suitable sign, $g_0-\varepsilon v_0^{0,\tau}$ has a simple zero $z(\varepsilon)\in(x_{j_0-1}, x_{j_0})$, $z(\varepsilon)\to x_{j_0}$ as $\varepsilon\to 0$. Therefore, it is clear that for all $\varepsilon\neq 0$, small enough and with a suitable sign, $g_0-\varepsilon v_0^{0,\tau}$ has a simple zero in $(J\backslash(x_{\imath_1}, x_{J_1}))\backslash\Omega$, which contradicts Claim 2. Thus, we conclude that g_0 has no zero interval in $[x_{J_1^- -\tau-1}, b)$, and so the claim is proved.

From Claim 3, g_0 does not go to 0 to the right. Then Lemma 4.23(a) implies that there exists μ large enough such that g_0 has no double zero in $[x_{J_\mu}, b)$, and

$$\{g_0 = 0\} \cap [x_{J_\mu}, b) = \Omega \cap [x_{J_\mu}, b).$$

As $v_0^{0,\tau}$ does not go to 0 to the right, (a$_5$) implies that $v_0^{0,\tau}$ has no double zero, and

$$\{v_0^{0,\tau} = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

From Claim 2, for all ε sufficiently small, $g_0 - \varepsilon v_0^{0,\tau}$ has no double zero and

$$\{g_0 - \varepsilon v_0^{0,\tau} = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

Thus, it is clear that Lemma 4.28(a) applies to g_0 and $v_0^{0,\tau}$ to conclude that g_0 does not have a lower right level than $v_0^{0,\tau}$. Therefore, (c) is proved. □

We are now in a position to prove Theorem 4.30.

Proof of Theorem 4.30. Assume that $v_0^{0,\tau}$ does not go to 0 to the right. Then Theorem 4.26 shows that there exists $\lim_{x \uparrow b} (g(x)/v_0^{0,\tau}(x))$, $x \notin \Omega$, for all $g \in \Psi(\mathscr{S}_1)$. It is obvious that $\mathscr{S}_1^{0,\tau+1}$ becomes a linear subspace of $\mathscr{S}_1^{0,\tau}$. Let

$$\theta_l^{0,\tau} := \lim_{\substack{x \uparrow b \\ x \notin \Omega}} \frac{v_l^{0,\tau}(x)}{v_0^{0,\tau}(x)}, \quad l = 1, 2, \ldots, m_1 - \tau - 1.$$

Then it follows from (a) in Lemma 4.32 that $0 < |\theta_l^{0,\tau}| < \infty$, $l = 1, 2, \ldots, m_1 - \tau - 1$. Therefore, $v_l^{0,\tau} - \theta_l^{0,\tau} v_0^{0,\tau}$ has a lower right level than $v_0^{0,\tau}$. Moreover, it is not difficult to see that $\{V_l^{0,\tau} - \theta_l^{0,\tau} V_0^{0,\tau}\}_{l=1}^{m_1-\tau-1}$ is linearly independent. Then $\mathscr{S}_1^{0,\tau+1}$ is precisely the $(m_1 - \tau - 1)$-dimensional space spanned by this basis. We now show that $\mathscr{S}_1^{0,\tau+1}$ is a WT-space. As every G in $\mathscr{S}_1^{0,\tau+1}$ is also in the $(m_1 - \tau)$-dimensional WT-space $\mathscr{S}_1^{0,\tau}$, G has at most $m_1 - \tau - 1$ sign changes. On the other hand, $\Psi(G)$ has a lower right level than $v_0^{0,\tau}$. Thus using (b) in Lemma 4.32 we conclude that G has at most $m_1 - \tau - 2$ sign changes. So $\mathscr{S}_1^{0,\tau+1}$ is weak Chebyshev. This proves (i).

To prove (ii), for each $l = 0, 1, \ldots, m_1 - \tau - 2$ we use Property [JKZ] in the $(m_1 - \tau - 1)$-dimensional WT-space $\mathscr{S}_1^{0,\tau+1}$ to obtain a $V_l^{0,\tau+1} \in \mathscr{S}_1^{0,\tau+1} \backslash \{0\}$ for which $\{y_k^l\}_{k=1}^{m_1-\tau-2}$ is an alternating set. Observe that $\{y_k^l\}_{k=1}^{m_1-\tau-2}$ is in $(x_{\iota_1^+}, x_{J_1^- - \tau - 1})$ (Remark 4.29).

We claim that for each $l = 0, 1, \ldots, m_1 - \tau - 2$, $V_l^{0,\tau+1}$ has no zero interval in $[x_{\iota_1}, x_{J_1^- - \tau - 1}]$. Assume to the contrary that for some l', $V_{l'}^{0,\tau+1}$ has a zero interval in $[x_{\iota_1}, x_{J_1^- - \tau - 1}]$. Then the location of the points $y_1^{l'}, y_2^{l'}, \ldots, y_{m_1-\tau-2}^{l'}$ implies that $[x_{\iota_1}, x_{J_1^- - \tau - 1}]$ has to be a zero interval of $V_{l'}^{0,\tau+1}$. Under this condition, and taking into account that $V_{l'}^{0,\tau+1}$ is in $\mathscr{S}_1^{0,\tau}$, we deduce from (c) of Lemma 4.32 that $v_{l'}^{0,\tau+1}$ has the same right level as $v_0^{0,\tau}$, which contradicts the definition of $V_{l'}^{0,\tau+1}$. This proves the claim. Therefore, each $V_l^{0,\tau+1}$ changes sign at $y_1^l, y_2^l, \ldots, y_{m_1-\tau-2}^l$. As the restriction of every $V_l^{0,\tau+1}$ to $[x_{\imath_1^+}, x_{J_1^-{-}\tau-1}]$ is in $\mathscr{S}_{\imath_1^+, J_1^- -\tau-1}$, it follows from Lemma 4.11(a) that those points are the only zeros (of multiplicity 1) of $V_l^{0,\tau+1}$ in $[x_{\imath_1^+}, x_{J_1^- -\tau-1}]$. Then it is easy to see that the set $\{V_l^{0,\tau+1}\}_{l=0}^{m_1-\tau-2}$, obtained by the replacement method based on $\{y_k\}_{k=1}^{m_1-\tau-2}$, is linearly independent. Therefore, $\{V_l^{0,\tau+1}\}_{l=0}^{m_1-\tau-2}$ is a basis for $\mathscr{S}_1^{0,\tau+1}$ satisfying that for $l = 0, 1, \ldots, m_1 - \tau - 2$, the spline $V_l^{0,\tau+1}$ changes sign at the $m_1 - \tau - 2$ points $y_1^l, y_2^l, \ldots, y_{m_1-\tau-2}^l$.

We now show that for $l = 0, 1, \ldots m_1 - \tau - 2$, the spline $V_l^{0,\tau+1}$ has no zero interval in $[x_{\imath_1}, x_{J_1}]$. We have seen above that for each l, $V_l^{0,\tau+1}$ has no zero interval in $[x_{\imath_1}, x_{J_1^- -\tau-1}]$. Suppose that for some l, $V_l^{0,\tau+1}$ has a zero interval $[x_{i_0}, x_{i_0+1}]$, $J_1^- - \tau - 1 \leqslant i_0 \leqslant J_1 - 1$. Then $V_l^{0,\tau+1}|_{[x_{\imath_1^+}, x_{i_0}]}$ is in the $(i_0 - \imath_1^+)$-dimensional WT-space $\mathscr{S}^-_{\imath_1^+, i_0}$. On the other hand, $V_l^{0,\tau+1}$ changes sign at the $m_1 - \tau - 2$ points $y_1^l, y_2^l, \ldots, y_{m_1-\tau-2}^l$, and by hypothesis, $\tau < n - 1 - \sigma(J_1)$. Therefore,

$$
\begin{aligned}
m_1 - \tau - 2 &= J_1^- - \imath_1^+ + n - 1 - \tau - 2 \\
&> J_1^- - \imath_1^+ + n - 1 - (n - 1 - \sigma(J_1)) - 2 \\
&= J_1 - \sigma(J_1) - \imath_1^+ + \sigma(J_1) - 2 \\
&\geqslant i_0 - \imath_1^+ - 1.
\end{aligned}
$$

This contradicts that $\mathscr{S}^-_{\imath_1^+, i_0}$ is an $(i_0 - \imath_1^+)$-dimensional WT-space. Thus, for $l = 0, 1, \ldots, m_1 - \tau - 2$, the spline $V_l^{0,\tau+1}$ has no zero interval. So to prove that $v_l^{0,\tau+1}$ has no zero interval in $(a, x_{J_1}]$ it is sufficient to show that $v_l^{0,\tau+1}$ has no zero interval in $(a, x_{\imath_1}]$. According to (a$_1$), to achieve this result, and also to complete the proof of (a$_1'$), we shall prove that $v_l^{0,\tau+1}$ is proportional to $v_l^{0,\tau}$, and so to v_l as well, on $(a, x_{J_1^- -\tau-1}]$. Observe that for each $l = 0, 1, \ldots, m_1 - \tau - 2$, the $m_1 - \tau - 2$ points $y_1^l, y_2^l, \ldots, y_{m_1-\tau-2}^l$ are simple zeros of $V_l^{0,\tau+1}|_{[x_{\imath_1^+}, x_{J_1^- -\tau-1}]}$ and also of $V_l^{0,\tau}|_{[x_{\imath_1^+}, x_{J_1^- -\tau-1}]}$, both in the space $\mathscr{S}_{\imath_1^+, J_1^- -\tau-1}$. Hence, using Lemma 4.11(a) we see that there exists a constant, say $\lambda_l \neq 0$, such that $V_l^{0,\tau+1} = \lambda_l V_l^{0,\tau}$ on $[x_{\imath_1}, x_{J_1^- -\tau-1}]$. Then applying Theorem 4.19 to $V_l^{0,\tau+1} - \lambda_l V_l^{0,\tau}$ we indeed deduce that $v_l^{0,\tau+1} = \lambda_l v_l^{0,\tau}$ on $(a, x_{J_1^- -\tau-1}]$. This proves (a$_1'$).

To prove (a$_2'$), let $G \in \mathscr{S}_1^{0,\tau+1}$. As $\mathscr{S}_1^{0,\tau+1}$ is contained in $\mathscr{S}_1^{0,\tau}$, applying (a$_2$) to $G \in \mathscr{S}_1^{0,\tau}$ we see that for every $v \in \mathbb{N}$, g $(= \Psi(G))$ has at most $m_v - \tau - 1$ sign changes on $(x_{\imath_v}, x_{J_v})$. So to complete the proof of (a$_2'$) it is sufficient to show that for any $v \in \mathbb{N}$, g does not have $m_v - \tau - 1$ sign changes on $(x_{\imath_v}, x_{J_v})$. Suppose to the contrary that for some $\mu \in \mathbb{N}$, g has $m_\mu - \tau - 1$ sign changes on $(x_{\imath_\mu}, x_{J_\mu})$. Then (b) in Lemma 4.23 shows that g has the same right level as $v_0^{0,\tau}$, which contradicts that G is in $\mathscr{S}_1^{0,\tau+1}$. This proves (a$_2'$).

To prove (a_3'), let $G \in \mathscr{S}_1^{0,\tau+1} \subset \mathscr{S}_1^{0,\tau}$ and assume that for some $\mu \in \mathbb{N}$, the spline g ($= \Psi(G)$) has $m_\mu - (\tau+1) - 2$ sign changes on $(x_{\imath_\mu}, x_{J_\mu})$. Suppose also that g does not go to 0 to the right nor to the left and that $g^{(n-2)}$ has no zero in some interval $[x_{i_1}, x_{i_1+1}]$. As g does not go to 0 to the right nor to the left, Lemma 4.23 implies that g has no zero interval in $J \backslash (x_{\imath_\mu}, x_{J_\mu})$. Consequently, to prove (a_3') it suffices to show that g has no double zeros and that g has no simple zero at points in $(J \backslash (x_{\imath_\mu}, x_{J_\mu})) \backslash \Omega$. We shall use the following facts. As $v_0^{0,\tau}$ does not go to 0 to the right (nor to the left), (a_5) implies that $v_0^{0,\tau}$ has no zero interval and that all the zeros of $v_0^{0,\tau}$ have multiplicity one. Furthermore, for all ε sufficiently small, $g - \varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left. Finally, as $g^{(n-2)}$ has no zero in $[x_{i_1}, x_{i_1+1}]$, the spline $(g - \varepsilon v_0^{0,\tau})^{(n-2)}$ has no zero in $[x_{i_1}, x_{i_1+1}]$ for all ε sufficiently small. Then we can apply Lemma 4.24 with $V_0^{0,\tau}$ in the place of V.

Suppose that z' is a double zero of g and $z' \in (J \backslash \Omega) \cup (x_{\imath_\mu}, x_{J_\mu})$. Choose a $\kappa > \mu$ for which $z' \in (x_{\imath_\kappa}, x_{J_\kappa})$. Then it follows from Lemma 4.24(a) (with $V_0^{0,\tau}$ in the place of V) that $g - \varepsilon v_0^{0,\tau}$ has at least $m_\kappa - (\tau+1) - 2 + 2 = m_\kappa - \tau - 1$ sign changes on $(x_{\imath_\kappa}, x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign. From (a_2) applied to $G - \varepsilon V_0^{0,\tau}$, in $\mathscr{S}_1^{0,\tau}$, we see that $g - \varepsilon v_0^{0,\tau}$ has at most $m_\nu - \tau - 1$ sign changes on $(x_{\imath_\nu}, x_{J_\nu})$ for each $\nu \in \mathbb{N}$. Therefore, the spline $g - \varepsilon v_0^{0,\tau}$ has $m_\kappa - \tau - 1$ sign changes on $(x_{\imath_\kappa}, x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign. Accordingly, we can choose an ε sufficiently small and with a suitable sign in such a way that $g - \varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left, $g - \varepsilon v_0^{0,\tau}$ has $m_\kappa - \tau - 1$ sign changes on $(x_{\imath_\kappa}, x_{J_\kappa})$, and $(g - \varepsilon v_0^{0,\tau})^{(n-2)}$ has no zero in $[x_{i_1}, x_{i_1+1}]$. This contradicts (a_4) applied to $G - \varepsilon V_0^{0,\tau} \in \mathscr{S}_1^{0,\tau}$. Thus, g has no double zero at points in $(J \backslash \Omega) \cup (x_{\imath_\mu}, x_{J_\mu})$.

Suppose now that $z'' \in \Omega \cap (J \backslash (x_{\imath_\mu}, x_{J_\mu}))$ is a double zero of g. Pick a $\kappa > \mu$ for which $z'' \in (x_{\imath_\kappa}, x_{J_\kappa})$. Then from Lemma 4.24(b), $g - \varepsilon v_0^{0,\tau}$ has at least $m_\kappa - (\tau+1) - 2 + 1 = m_\kappa - \tau - 2$ sign changes on $(x_{\imath_\kappa} x_{J_\kappa})$ for all ε sufficiently small. Hence, applying (a_2) to $G - \varepsilon V_0^{0,\tau}$, in $\mathscr{S}_1^{0,\tau}$, we deduce that $g - \varepsilon v_0^{0,\tau}$ has $m_\kappa - \tau - t_\varepsilon$ sign changes on $(x_{\imath_\kappa}, x_{J_\kappa})$, $1 \leqslant t_\varepsilon \leqslant 2$, for all ε sufficiently small. Thus, we see that for all ε sufficiently small, the following conditions hold:

- $g - \varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left;
- $(g - \varepsilon v_0^{0,\tau})^{(n-2)}$ has no zero in $[x_{i_1}, x_{i_1+1}]$;
- $g - \varepsilon v_0^{0,\tau}$ has $m_\kappa - \tau - t_\varepsilon$ sign changes on $(x_{\imath_\kappa}, x_{J_\kappa})$, $1 \leqslant t_\varepsilon \leqslant 2$.

Hence, if $t_\varepsilon = 1$ for some ε satisfying the above conditions, then (a_4) applied to $G - \varepsilon V_0^{0,\tau} \in \mathscr{S}_1^{0,\tau}$ is contradicted. Thus, $t_\varepsilon = 2$ for all ε sufficiently small. Then for all ε sufficiently small, (a_3) applied to $G - \varepsilon V_0^{0,\tau} \in \mathscr{S}_1^{0,\tau}$ shows that $g - \varepsilon v_0^{0,\tau}$ has no double zero in $J\backslash(x_{\iota_\kappa}, x_{J_\kappa})$, and

$$\{g - \varepsilon v_0^{0,\tau} = 0\} \cap (J\backslash(x_{\iota_\kappa}, x_{J_\kappa})) = \Omega \cap (J\backslash(x_{\iota_\kappa}, x_{J_\kappa})).$$

On the other hand, we apply Lemma 4.23 to g and assume that κ is so large that we deduce that all the zeros of g in $J\backslash(x_{\iota_\kappa}, x_{J_\kappa})$ have multiplicity one, and

$$\{g = 0\} \cap (J\backslash(x_{\iota_\kappa}, x_{J_\kappa})) = \Omega \cap (J\backslash(x_{\iota_\kappa}, x_{J_\kappa})).$$

Furthermore, as $v_0^{0,\tau}$ does not go to 0 to the right, (a_5) applied to $V_0^{0,\tau}$ implies that all the zeros of $v_0^{0,\tau}$ have multiplicity one, and

$$\{v_0^{0,\tau} = 0\} \cap J\backslash(x_{\iota_1}, x_{J_1}) = \Omega \cap J\backslash(x_{\iota_1}, x_{J_1}).$$

We are now in a position to apply (c) in Lemma 4.28 to show that g does not have a lower right level than $v_0^{0,\tau}$. But this contradicts that G is in $\mathscr{S}_1^{0,\tau+1}$. Consequently, g has no double zero at points in $\Omega \cap (J\backslash(x_{\iota_\mu}, x_{J_\mu}))$.

We have just proven that g has no double zero. So to prove (a_3') it remains to show that g has no simple zero at points in $(J\backslash(x_{\iota_\mu}, x_{J_\mu}))\backslash\Omega$. This proof follows in a similar way to that of the previous case. So (a_3') is proved.

We now prove (a_4'). Let $G \in \mathscr{S}_1^{0,\tau+1}$ and assume that for some $\mu \in \mathbb{N}$, the restriction of g $(= \Psi(G))$ to $(x_{\iota_\mu}, x_{J_\mu})$ has $m_\mu - (\tau + 1) - 1$ sign changes. Suppose also that g does not go to 0 to the right nor to the left. We shall apply Lemmas 4.24 and 4.28 to g and $v_0^{0,\tau}$. For this we shall use, according to (a_5), that $v_0^{0,\tau}$ has no zero interval, all the zeros of $v_0^{0,\tau}$ have multiplicity one, and

$$\{v_0^{0,\tau} = 0\} \cap (J\backslash(x_{\iota_1}, x_{J_1})) = \Omega \cap (J\backslash(x_{\iota_1}, x_{J_1})).$$

We now prove that g has no double zero, and $\{g = 0\} \cap (J\backslash(x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J\backslash(x_{\iota_\mu}, x_{J_\mu}))$. As g does not go to 0 to the right nor to the left, Lemma 4.23 implies that g has no zero interval in $J\backslash(x_{\iota_\mu}, x_{J_\mu})$. So it is sufficient to prove that g has no double zero and that g has no simple zeros in $(J\backslash(x_{\iota_\mu}, x_{J_\mu}))\backslash\Omega$. To prove this observe that if g has a double zero, or a simple zero in $(J\backslash(x_{\iota_\mu}, x_{J_\mu}))\backslash\Omega$, then Lemma 4.24, in conjunction with (a_2), shows that for each $\kappa > \mu$ and large enough, $g - \varepsilon v_0^{0,\tau}$ has $m_\kappa - \tau - 1$ sign changes on $(x_{\iota_\kappa}, x_{J_\kappa})$ for all ε sufficiently small. Now the proof follows in a similar way as that of (a_3'), and we again deduce the contradiction that g does not have a lower right level than $v_0^{0,\tau}$. Thus, g has no double zero, and

$$\{g = 0\} \cap (J\backslash(x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J\backslash(x_{\iota_\mu}, x_{J_\mu})).$$

To prove that $g^{(n-2)}$ has at least a zero in $[x_i, x_{i+1}]$, all $i \in \mathbb{Z}$, suppose to the contrary that there exists $i_2 \in \mathbb{Z}$ such that $g^{(n-2)}$ has no zero in $[x_{i_2}, x_{i_2+1}]$. Then since g has $m_\mu - \tau - 2$ sign changes on $(x_{\iota_\mu}, x_{J_\mu})$, it is easy to see that for all ε sufficiently small the following conditions hold:

- $g - \varepsilon v_0^{0,\tau}$ does not go to 0 to the right nor to the left;

- $(g - \varepsilon v_0^{0,\tau})^{(n-2)}$ has no zero in $[x_{i_2}, x_{i_2+1}]$;
- $g - \varepsilon v_0^{0,\tau}$ has $m_\mu - \tau - t_\varepsilon$ sign changes on $(x_{\iota_\mu}, x_{J_\mu})$, $1 \leqslant t_\varepsilon \leqslant 2$.

We now apply (a$_4$) to deduce that $t_\varepsilon = 2$ necessarily, whence (a$_3$) implies that $g - \varepsilon v_0^{0,\tau}$ has no double zero, and

$$\{g - \varepsilon v_0^{0,\tau} = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu}))$$

for all ε sufficiently small. Then observe that we can apply Lemma 4.28(c) to g and $v_0^{0,\tau}$ to conclude that g does not have a lower right level than $v_0^{0,\tau}$, which is a contradiction. Thus, $g^{(n-2)}$ has at least a zero in $[x_i, x_{i+1}]$, all $i \in \mathbb{Z}$. This completes the proof of (a$'_4$).

We now prove (a$'_5$). From (a$'_1$), $v_l^{0,\tau+1}$ has no zero interval in $(a, x_{J_1}]$. Furthermore, if $v_l^{0,\tau+1}$ does not go to 0 to the right, then Lemma 4.23(a) implies that $v_l^{0,\tau+1}$ has no zero interval in $[x_{J_1}, b)$. Therefore, $v_l^{0,\tau+1}$ has no zero interval. So the first statement in (a$'_5$) is proved.

From (a$'_1$), $V_l^{0,\tau+1}$ has exactly $m_1 - \tau - 2$ (simple) zeros in (x_{ι_1}, x_{J_1}). So we can apply (a$'_4$) to $V_l^{0,\tau+1} \in \mathscr{S}_1^{0,\tau+1}$, since $v_l^{0,\tau+1}$ does not go to 0 to the right nor to the left. Then $v_l^{0,\tau+1}$ has no double zero, and

$$\{v_l^{0,\tau+1} = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

Thus, counting the points of $\Omega \cap ((x_{\iota_\kappa}, x_{J_\kappa}) \backslash (x_{\iota_1}, x_{J_1}))$, $\kappa > 1$, we deduce that $v_l^{0,\tau+1}$ has $m_\kappa - \tau - 2$ simple zeros in $(x_{\iota_\kappa}, x_{J_\kappa})$, $\kappa \geqslant 1$. To complete the proof of (a$'_5$) it remains to show that $v_l^{0,\tau+1}$ has no zero of odd multiplicity greater than two. Assume to the contrary that $v_l^{0,\tau+1}$ has a zero of odd multiplicity greater than two at a point $z \in (x_{\iota_\kappa}, x_{J_\kappa})$, $\kappa \geqslant 1$. From (a$_5$), all the zeros of $v_l^{0,\tau}$ have multiplicity one, and

$$\{v_l^{0,\tau} = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

Using also (a$_1$) and (a$'_1$) we see that all the zeros of $v_l^{0,\tau+1}$ are zeros of $v_l^{0,\tau}$ as well. Accordingly, z is a zero of $v_l^{0,\tau}$ of multiplicity one. Therefore, it is an easy exercise to prove that for all ε sufficiently small and with a suitable sign, $v_l^{0,\tau+1} - \varepsilon v_l^{0,\tau}$ has at least three different simple zeros, say $z_1(\varepsilon)$, $z_2(\varepsilon)$ and $z_3(\varepsilon)$, satisfying $z_1(\varepsilon) \uparrow z$ as $\varepsilon \to 0$, $z_2(\varepsilon) = z$ and $z_3(\varepsilon) \downarrow z$ as $\varepsilon \to 0$. Then for all $\varepsilon \neq 0$, small enough and with a suitable sign, the spline $v_l^{0,\tau+1} - \varepsilon v_l^{0,\tau}$ has at least $m_\kappa - \tau - 2 + 2 = m_\kappa - \tau$ simple zeros in $(x_{\iota_\kappa}, x_{J_\kappa})$. This contradicts (a$_2$) applied to $V_l^{0,\tau+1} - \varepsilon V_l^{0,\tau} \in \mathscr{S}_1^{0,\tau}$. Thus, all the zeros of $v_l^{0,\tau+1}$ have multiplicity one. This proves (a$'_5$), which completes the proof of (ii). □

Lemma 4.33. *For each $l = 0, 1, \ldots, m_1 - 1$, let $V_l^{0,0} := V_l$. Then the m_1-dimensional WT-space $\mathscr{S}_1^{0,0}$ $(= \mathscr{S}_1)$ and the basis $\{V_0^{0,0}, V_1^{0,0}, \ldots, V_{m_1-1}^{0,0}\}$ satisfy properties* (a$_1$)–(a$_5$) *of Theorem* 4.30 *with* $\tau = 0$.

Proof. By the construction of V_l in Remark 4.29, $l = 0, 1, \ldots, m_1 - 1$, the spline $V_l^{0,0}$ ($= V_l$) changes sign at the $m_1 - 1$ points $y_1^l, y_2^l, \ldots, y_{m_1-1}^l$. So $V_l^{0,0}$ has $m_1 - 1$ sign changes, since $V_l^{0,0}$ is in the m_1-dimensional WT-space $\mathscr{S}_1$ ($= \mathscr{S}_1^{0,0}$). Lemma 4.21 now implies that $v_l^{0,0}$ ($= v_l$) has no zero interval, and thus it has no zero interval in $(a, x_{J_1}]$. This proves (a_1) with $\tau = 0$.

Observe that (a_2) holds with $\tau = 0$ because the restriction of g ($= \Psi(G)$) to (x_{ι_v}, x_{J_v}) is in the m_v-dimensional WT-space $\mathscr{S}_v$.

To prove (a_3), let $G \in \mathscr{S}_1^{0,0}$ and suppose that for some $\mu \in \mathbb{N}$, the restriction of g ($= \Psi(G)$) to $(x_{\iota_\mu}, x_{J_\mu})$ has $m_\mu - 2$ sign changes. We are also assuming that g does not go to 0 to the right nor to the left and that the broken line $g^{(n-2)}$ has no zero in some interval $[x_{i_1}, x_{i_1+1}]$. Theorem 4.19 implies that

$$\{g = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) \supseteq \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

On the other hand, as g does not go to 0 to the right nor to the left, Lemma 4.23 implies that g has no zero interval in $J \backslash (x_{\iota_1}, x_{J_1})$. In this way, to see that g has no double zero, and that

$$\{g = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})),$$

it is sufficient to demonstrate that g has no double zero, and that g has no simple zero in $(J \backslash (x_{\iota_\mu}, x_{J_\mu})) \backslash \Omega$. As $V_0^{0,0}$ has $m_1 - 1$ sign changes, Lemma 4.21 implies that $v_0^{0,0}$ has no zero interval and that all the zeros of $v_0^{0,0}$ have multiplicity one. Hence, we can apply Lemma 4.24 with $V = V_0^{0,0}$. Suppose that g has a double zero, or a simple zero at a point in $(J \backslash (x_{\iota_\mu}, x_{J_\mu})) \backslash \Omega$. Then Lemma 4.24 (with $V_0^{0,0}$ in the place of V and with $t = 2$) implies that in either case, for a κ large enough, $g - \varepsilon v_0^{0,0}$ has $m_\kappa - 1$ sign changes on $(x_{\iota_\kappa}, x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign. Then from Lemma 4.21, $(g - \varepsilon v_0^{0,0})^{(n-2)}$ has a simple zero in (x_i, x_{i+1}), all $i \in \mathbb{Z}$. But, as $g^{(n-2)}$ has no zero in $[x_{i_1}, x_{i_1+1}]$, for all ε sufficiently small $(g - \varepsilon v_0^{0,0})^{(n-2)}$ has no zero in $[x_{i_1}, x_{i_1+1}]$. So we obtain a contradiction. In consequence, g has no double zero, and g has no simple zero in $(J \backslash (x_{\iota_\mu}, x_{J_\mu})) \backslash \Omega$, which proves ($a_3$) with $\tau = 0$.

Finally, note that (a_4) and (a_5), with $\tau = 0$, follow from Lemma 4.21. □

Due to Remark 4.29, $V_0^{0,0}$ ($= V_0$) has $m_1 - 1$ sign changes. Then by Lemma 4.21, $v_0^{0,0}$ has no zero interval, and therefore $v_0^{0,0}$ does not go to 0 to the right (nor to the left). From Lemma 4.33, $\mathscr{S}_1^{0,0}$ and the basis $\{V_0^{0,0}, V_1^{0,0}, \ldots, V_{m_1-1}^{0,0}\}$ satisfy properties (a_1)–(a_5) of Theorem 4.30. We thus apply Theorem 4.30 to the space $\mathscr{S}_1^{0,0}$ to get the $(m_1 - 1)$-dimensional WT-subspace $\mathscr{S}_1^{0,1}$ and the basis $\{V_0^{0,1}, V_1^{0,1}, \ldots, V_{m_1-2}^{0,1}\}$, which fulfill ($a_1$)–($a_5$) of Theorem 4.30 with $\tau = 1$. If $1 < n - 1 - \sigma(J_1)$ and $v_0^{0,1}$ ($= \Psi(V_0^{0,1})$) does not go to 0 to the right, then Theorem 4.30 again applies to get the $(m_1 - 2)$-dimensional WT-subspace $\mathscr{S}_1^{0,2}$ and the basis $\{V_0^{0,2}, V_1^{0,2}, \ldots, V_{m_1-3}^{0,2}\}$, which fulfill

(a_1)–(a_5) of Theorem 4.30 with $\tau = 2$. Proceeding in this way, if

$$\tau^* := 1 + \max_{0 \leqslant \tau < n-1-\sigma(J_1)} \{\tau : v_0^{0,\tau} (= \Psi(V_0^{0,\tau})) \text{ does not go to 0 to the right}\},$$

then it is clear that we obtain the WT-subspaces

$$(\mathscr{S}_1 =) \mathscr{S}_1^{0,0} \supset \mathscr{S}_1^{0,1} \supset \cdots \supset \mathscr{S}_1^{0,\tau^*},$$

with basis

$$\{V_0^{0,0}, \ldots, V_{m_1-1}^{0,0}\}, \{V_0^{0,1}, \ldots, V_{m_1-2}^{0,1}\}, \ldots, \{V_0^{0,\tau^*}, \ldots, V_{m_1-\tau^*-1}^{0,\tau^*}\},$$

respectively. As an immediate consequence of this discussion, we have the following result.

Corollary 4.34. *For* $0 \leqslant \tau \leqslant \tau^*$, *the set* $\mathscr{S}_1^{0,\tau}$ *is an* $(m_1 - \tau)$*-dimensional WT-subspace of* $\mathscr{S}_1$. *Also,* $\mathscr{S}_1^{0,\tau}$ *and the basis* $\{V_0^{0,\tau}, V_1^{0,\tau}, \ldots, V_{m_1-\tau-1}^{0,\tau}\}$ *fulfill properties* (a_1)–(a_5) *of Theorem* 4.30 *as well as Proposition* 4.31 *and Lemma* 4.32.

In the sequel we shall use Corollary 4.34 without an explicit reference whenever we apply (a_1)–(a_5) of Theorem 4.30, Proposition 4.31 or Lemma 4.32 to the spaces $\mathscr{S}_1^{0,\tau}$, $0 \leqslant \tau \leqslant \tau^*$.

Remark 4.35. It is clear that $v_0^{0,\tau}$ does not go to 0 to the right if $0 \leqslant \tau < \tau^*$. On the contrary, we now assert that v_0^{0,τ^*} $(= \Psi(V_0^{0,\tau^*}))$ goes to 0 to the right. Indeed, this follows immediately from the definition of τ^* whenever $\tau^* < n - 1 - \sigma(J_1)$. So consider $\tau^* = n - 1 - \sigma(J_1)$. From Theorem 4.19 we deduce that $\Psi(G)$ goes to 0 to the right for all $G \in \mathscr{S}^-_{\iota_1,J_1}$. Hence, $\mathscr{S}^-_{\iota_1,J_1} \cap \mathscr{S}_1 \subseteq \mathscr{S}_1^{0,\tau^*}$.

Furthermore, $\dim(\mathscr{S}^-_{\iota_1,J_1} \cap \mathscr{S}_1) = m_1 - (n - 1 - \sigma(J_1)) = m_1 - \tau^* = \dim \mathscr{S}_1^{0,\tau^*}$. Thus, $\mathscr{S}_1^{0,\tau^*} = \mathscr{S}^-_{\iota_1,J_1} \cap \mathscr{S}_1$ whenever $\tau^* = n - 1 - \sigma(J_1)$, and therefore v_0^{0,τ^*} goes to 0 to the right. This proves the assertion. Finally, using (a) in Lemma 4.32 we conclude that for $l = 0, 1, \ldots, m_1 - \tau - 1$, the spline $v_l^{0,\tau}$ goes to 0 to the right if and only if $\tau = \tau^*$.

Now let τ be fixed, $0 \leqslant \tau < \tau^*$, and consider the $(m_1 - \tau)$-dimensional WT-space $\mathscr{S}_1^{0,\tau}$. In the following theorem we shall prove the existence of WT-subspaces of $\mathscr{S}_1^{0,\tau}$ "by lowering the left level". In particular, the theorem establishes five properties for $\mathscr{S}_1^{0,\tau}$. Lemma 4.39 shows that $\mathscr{S}_1^{0,\tau}$ indeed satisfies these properties. Note that if $\tau = 0$, then Theorem 4.36 is analogous to Theorem 4.30.

Theorem 4.36. *Let* $\mathscr{S}_1^{\rho,\tau}$ *be an* $(m_1 - \tau - \rho)$*-dimensional WT-subspace of* $\mathscr{S}_1^{0,\tau}$, $0 \leqslant \tau < \tau^*$, $0 \leqslant \rho < n - 1 - \sigma(\iota_1)$, *with a basis* $\{V_\rho^{\rho,\tau}, V_{\rho+1}^{\rho,\tau}, \ldots, V_{m_1-\tau-1}^{\rho,\tau}\}$, *and such that* *the following properties hold*:

(b_1) *For each* $l = \rho, \rho+1, \dots, m_1 - \tau - 1$, *the spline* $V_l^{\rho,\tau}$ *changes sign at the* $m_1 - \tau - \rho - 1$ *points* $y^l_{\rho+1}, y^l_{\rho+2}, \dots, y^l_{m_1-\tau-1}$, *whence it has* $m_1 - \tau - \rho - 1$ *sign changes. Moreover, for each* $l = \rho, \rho+1, \dots, m_1 - \tau - 1$, $v_l^{\rho,\tau} := \Psi(V_l^{\rho,\tau})$ *is proportional to* $v_l^{0,\tau}$ *on* $[x_{\iota_1^+ + \rho}, b)$, *and* $v_l^{\rho,\tau}$ *has no zero interval in* $[x_{\iota_1}, b)$.

(b_2) *Let* $G \in \mathscr{S}_1^{\rho,\tau}$. *Then for every* $v \in \mathbb{N}$, *the restriction of* $g := \Psi(G)$ *to* (x_{ι_v}, x_{J_v}) *has at most* $m_v - \tau - \rho - 1$ *sign changes.*

(b_3) *Let* $G \in \mathscr{S}_1^{\rho,\tau}$ *and assume that for some* $\mu \in \mathbb{N}$, *the restriction of* $g\ (= \Psi(G))$ *to* $(x_{\iota_\mu}, x_{J_\mu})$ *has* $m_\mu - \tau - \rho - 2$ *sign changes. If g does not go to* 0 *to the right nor to the left and the broken line* $g^{(n-2)}$ *has no zero in some interval* $[x_{j_1}, x_{j_1+1}]$, *then g has no double zero, and*

$$\{g = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})).$$

(b_4) *Let* $G \in \mathscr{S}_1^{\rho,\tau}$ *and assume that for some* $\mu \in \mathbb{N}$, *the restriction of* $g\ (= \Psi(G))$ to $(x_{\iota_\mu}, x_{J_\mu})$ *has* $m_\mu - \tau - \rho - 1$ *sign changes. Suppose also that g does not go to* 0 *to the right nor to the left. Then g has no double zero, and*

$$\{g = 0\} \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})) = \Omega \cap (J \backslash (x_{\iota_\mu}, x_{J_\mu})).$$

Moreover, $g^{(n-2)}$ *has at least a zero in* $[x_j, x_{j+1}]$, *all* $j \in \mathbb{Z}$.

(b_5) *For each* $l = \rho, \rho+1, \dots, m_1 - \tau - 1$, *if* $v_l^{\rho,\tau}\ (= \Psi(V_l^{\rho,\tau}))$ *does not go to* 0 *to the left, then* $v_l^{\rho,\tau}$ *has no zero interval, all the zeros of* $v_l^{\rho,\tau}$ *have multiplicity one, and*

$$\{v_l^{\rho,\tau} = 0\} \cap (J \backslash (x_{\iota_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\iota_1}, x_{J_1})).$$

If $v_\rho^{\rho,\tau}$ *does not go to 0 to the left, then*

(i) *The set* $\mathscr{S}_1^{\rho+1,\tau} := \{G \in \mathscr{S}_1^{\rho,\tau} : \Psi(G)$ *has a lower left level than* $v_\rho^{\rho,\tau}\}$ *is an* $(m_1 - \tau - \rho - 1)$*-dimensional WT-subspace of* $\mathscr{S}_1^{\rho,\tau}$.

(ii) *There exists a basis* $\{V_{\rho+1}^{\rho+1,\tau}, V_{\rho+2}^{\rho+1,\tau}, \dots, V_{m_1-\tau-1}^{\rho+1,\tau}\}$ *for* $\mathscr{S}_1^{\rho+1,\tau}$ *such that* $\mathscr{S}_1^{\rho+1,\tau}$ *and this basis fulfill the analogs of* (b_1)–(b_5) *with* $\rho+1$ *in place of* ρ, *say* (b_1')–(b_5'), *respectively.*

The proofs of Proposition 4.37 and Lemma 4.38 are analogous to that of Proposition 4.31 and Lemma 4.32, respectively, and so we omit them. We only remark that $V_0^{0,\tau}$ and $v_0^{0,\tau}$ must be here replaced with $V_\rho^{\rho,\tau}$ and $v_\rho^{\rho,\tau}$, respectively. Also, Lemma 4.23(b) and Lemma 4.28(b) have to be used in Lemma 4.38 instead of Lemma 4.23(a) and Lemma 4.28(a), respectively.

Proposition 4.37. *Let* $\mathscr{S}_1^{\rho,\tau}$ *be an* $(m_1 - \tau - \rho)$*-dimensional WT-subspace of* $\mathscr{S}_1^{0,\tau}$, $0 \leqslant \tau < \tau^*$, $0 \leqslant \rho \leqslant n - 1 - \sigma(\iota_1)$, *with a basis* $\{V_\rho^{\rho,\tau}, V_{\rho+1}^{\rho,\tau}, \dots, V_{m_1-\tau-1}^{\rho,\tau}\}$. *Assume that*

$\mathscr{S}_1^{\rho,\tau}$ and this basis satisfy (b$_1$) *of Theorem* 4.36, *and let $G_0 \in \mathscr{S}_1^{\rho,\tau}$. If $[x_{i'}, x_{J_1^- \tau}]$ is a zero interval of G_0, $\imath_1 \leqslant i' \leqslant \imath_1^+ + \rho$, then $\Psi(G_0)$ vanishes identically on $(a, x_{J_1^- - \tau}]$.*

Lemma 4.38. *Let $\mathscr{S}_1^{\rho,\tau}$ be an $(m_1 - \tau - \rho)$-dimensional WT-subspace of $\mathscr{S}_1^{0,\tau}$, $0 \leqslant \tau < \tau^*$, $0 \leqslant \rho \leqslant n - 1 - \sigma(\imath_1)$, with a basis $\{V_\rho^{\rho,\tau}, V_{\rho+1}^{\rho,\tau}, \dots, V_{m_1-\tau-1}^{\rho,\tau}\}$. If $\mathscr{S}_1^{\rho,\tau}$ and this basis satisfy* (b$_1$)–(b$_5$) *of Theorem* 4.36, *then there holds*

(a) *The splines $v_l^{\rho,\tau}$ $(= \Psi(V_l^{\rho,\tau}))$, $l = \rho, \rho + 1, \dots, m_1 - \tau - \rho - 1$, have the same left level. Hence, $v_\rho^{\rho,\tau}$ does not have a lower left level than $\Psi(G)$ for any $G \in \mathscr{S}_1^{\rho,\tau}$.*
(b) *Let $G_1 \in \mathscr{S}_1^{\rho,\tau}$ and suppose that $g_1 := \Psi(G_1)$ has $m_\mu - \tau - \rho - 1$ sign changes on $(x_{\imath_\mu}, x_{J_\mu})$ for some $\mu \in \mathbb{N}$. Then g_1 has the same left level as $v_\rho^{\rho,\tau}$.*
(c) *Let $G_0 \in \mathscr{S}_1^{\rho,\tau} \backslash \{0\}$, and suppose that $[x_{\imath_1^+ + \rho + 1}, x_{J_1^- - \tau}]$ is a zero interval of G_0. Then $\Psi(G_0)$ has the same left level as $v_\rho^{\rho,\tau}$.*

Proof of Theorem 4.36. Assume $v_\rho^{\rho,\tau}$ does not go to 0 to the left. Then Theorem 4.26 shows that there exists $\lim_{x \downarrow a} (g(x)/v_\rho^{\rho,\tau}(x))$, $x \notin \Omega$, for all $g \in \Psi(\mathscr{S}_1)$. It is clear that $\mathscr{S}_1^{\rho+1,\tau}$ becomes a linear subspace of $\mathscr{S}_1^{\rho,\tau}$. Let

$$\tilde{\theta}_l^{\rho,\tau} := \lim_{\substack{x \downarrow a \\ x \notin \Omega}} \frac{v_l^{\rho,\tau}(x)}{v_\rho^{\rho,\tau}(x)}, \quad l = \rho + 1, \rho + 2, \dots, m_1 - \tau - 1.$$

Then it follows from (a) in Lemma 4.38 that $0 < |\tilde{\theta}_l^{\rho,\tau}| < \infty$. So $v_l^{\rho,\tau} - \tilde{\theta}_l^{\rho,\tau} v_\rho^{\rho,\tau}$ has a lower left level than $v_\rho^{\rho,\tau}$. Moreover, it is easily seen that

$$\{V_l^{\rho,\tau} - \tilde{\theta}_l^{\rho,\tau} V_\rho^{\rho,\tau}\}_{l=\rho+1}^{m_1-\tau-1}$$

is linearly independent. Then $\mathscr{S}_1^{\rho+1,\tau}$ is the $(m_1 - \tau - \rho - 1)$-dimensional space spanned by this basis. We now show that $\mathscr{S}_1^{\rho+1,\tau}$ is a WT-space. As every G in $\mathscr{S}_1^{\rho+1,\tau}$ is also in the $(m_1 - \tau - \rho)$-dimensional WT-space $\mathscr{S}_1^{\rho,\tau}$, the spline G has at most $m_1 - \tau - \rho - 1$ sign changes. On the other hand, $\Psi(G)$ has a lower left level than $v_\rho^{\rho,\tau}$. Thus, using (b) in Lemma 4.38 we conclude that G has at most $m_1 - \tau - \rho - 2$ sign changes. Thus, $\mathscr{S}_1^{\rho+1,\tau}$ is weak Chebyshev. This proves (i).

To prove (ii), for each $l = \rho + 1, \rho + 2, \dots, m_1 - \tau - 1$, we use Property [JKZ] in the $(m_1 - \tau - \rho - 1)$-dimensional WT-space $\mathscr{S}_1^{\rho+1,\tau}$ to obtain the spline $V_l^{\rho+1,\tau} \in \mathscr{S}_1^{\rho+1,\tau} \backslash \{0\}$ for which $\{y_k^l\}_{k=\rho+2}^{m_1-\tau-1}$ is an alternating set. Observe that $\{y_k^l\}_{k=\rho+2}^{m_1-\tau-1}$ is in $(x_{\imath_1^+ + \rho + 1}, x_{J_1^- - \tau})$ (Remark 4.29).

We now claim that for each $l = \rho + 1, \rho + 2, \dots, m_1 - \tau - 1$, the spline $V_l^{\rho+1,\tau}$ has no zero interval in $[x_{\imath_1^+ + \rho + 1}, x_{J_1^- - \tau}]$. Suppose to the contrary that for some l', $V_{l'}^{\rho+1,\tau}$ has a zero interval in $[x_{\imath_1^+ + \rho + 1}, x_{J_1^- - \tau}]$. Then the location of the points $y_{\rho+2}^{l'}, y_{\rho+3}^{l'}, \dots, y_{m_1-\tau-1}^{l'}$ forces $[x_{\imath_1^+ + \rho + 1}, x_{J_1^- - \tau}]$ to be a zero interval of $V_{l'}^{\rho+1,\tau}$. Therefore, (c) in Lemma 4.38 applied to $V_{l'}^{\rho+1,\tau} \in \mathscr{S}_1^{\rho,\tau}$ implies that $\Psi(V_{l'}^{\rho+1,\tau})$ has the same left level than $v_\rho^{\rho,\tau}$, which contradicts the definition of $V_{l'}^{\rho+1,\tau}$. This proves the claim. Thus, for each $l = \rho+1, \rho+2, \ldots, m_1-\tau-1$, $V_l^{\rho+1,\tau}$ changes sign at the $m_1-\tau-\rho-2$ points $y_{\rho+2}^l, y_{\rho+3}^l, \ldots, y_{m_1-\tau-1}^l$. As the restriction of every $V_l^{\rho+1,\tau}$ to the interval $[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]$ is in $\mathscr{S}_{\imath_1^+ +\rho+1,\ \jmath_1^- -\tau}$, it follows from Lemma 4.11(a) that those points are the only zeros (of multiplicity 1) of $V_l^{\rho+1,\tau}$ in $[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]$. Then it is easy to see that the set $\{V_l^{\rho+1,\tau}\}_{l=\rho+1}^{m_1-\tau-1}$, obtained by the replacement method based on $\{y_k\}_{k=\rho+2}^{m_1-\tau-1}$, is linearly independent. Thus, $\{V_l^{\rho+1,\tau}\}_{l=\rho+1}^{m_1-\tau-1}$ is a basis for $\mathscr{S}_1^{\rho+1,\tau}$ satisfying that for $l = \rho+1, \rho+2, \ldots, m_1-\tau-1$, the spline $V_l^{\rho+1,\tau}$ changes sign at the $m_1-\tau-(\rho+1)-1 = m_1-\tau-\rho-2$ points $y_{\rho+2}^l, y_{\rho+3}^l, \ldots, y_{m_1-\tau-1}^l$.

We have just seen that for $l = \rho+1, \rho+2, \ldots, m_1-\tau-1$, $V_l^{\rho+1,\tau}$ has no zero interval in $[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]$. So to prove that $v_l^{\rho+1,\tau}$ has no zero interval in $[x_{\imath_1}, b)$ it is sufficient to show that $v_l^{\rho+1,\tau}$ has neither zero interval in $[x_{\imath_1}, x_{\imath_1^+ +\rho+1}]$ nor in $[x_{\jmath_1^- -\tau}, b)$. If $v_l^{\rho+1,\tau}$ has a zero interval in $[x_{\imath_1}, x_{\imath_1^+ +\rho+1}]$, then $v_l^{\rho+1,\tau}|_{[i_0,\jmath_1^- -\tau]}$ is in the WT-space $\mathscr{S}_{i_0,\jmath_1^- -\tau}^+$, with $\imath_1 < i_0 < \imath_1^+ + \rho + 1$, so that $\dim \mathscr{S}_{i_0,\jmath_1^- -\tau}^+ \leqslant \jmath_1^- - \tau - \imath_1 - 1$. Now, we know that $v_l^{\rho+1,\tau}$ has $m_1-\tau-\rho-2$ sign changes on $(x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau})$ $(\subset [x_{i_0}, x_{\jmath_1^- -\tau}])$, and using that $\rho < n-1-\sigma(\imath_1)$ we get $m_1-\tau-\rho-2 \geqslant \jmath_1^- - \tau - \imath_1 - 1$, which is a contradiction. On the other hand, note that from (a$_1$) and because $\tau < \tau^*$, if $v_l^{\rho+1,\tau}$ is proportional to $v_l^{0,\tau}$ on $[x_{\imath_1^+ +\rho+1}, b)$, then $v_l^{\rho+1,\tau}$ has no zero interval in $[x_{\jmath_1^- -\tau}, b)$ $(\subset [x_{\imath_1^+ +\rho+1}, b))$. Thus, we now show that for $l = \rho+1, \rho+2, \ldots, m_1-\tau-1$, $v_l^{\rho+1,\tau}$ is proportional to $v_l^{0,\tau}$ on $[x_{\imath_1^+ +\rho+1}, b)$. For this purpose, observe that for each $l = \rho+1, \rho+2, \ldots, m_1-\tau-1$, the $m_1-\tau-\rho-2$ points $y_{\rho+2}^l, y_{\rho+3}^l, \ldots, y_{m_1-\tau-1}^l$ are simple zeros of $V_l^{\rho+1,\tau}|_{[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]}$ and also of $V_l^{\rho,\tau}|_{[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]}$, both in the space $\mathscr{S}_{\imath_1^+ +\rho+1,\jmath_1^- -\tau}$. Hence, applying Lemma 4.11(a) we easily deduce that $V_l^{\rho+1,\tau}|_{[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]}$ and $V_l^{\rho,\tau}|_{[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]}$ must be proportional. Then there is a linear combination of $V_l^{\rho+1,\tau}$ and $V_l^{\rho,\tau}$, which is in $\mathscr{S}_1^{\rho,\tau} \subset \mathscr{S}_1^{0,\tau}$, vanishing identically on $[x_{\imath_1^+ +\rho+1}, x_{\jmath_1^- -\tau}]$. Now, it is easy to deduce from Proposition 4.31 that $v_l^{\rho+1,\tau}$ and $v_l^{\rho,\tau}$ are proportional on $[x_{\imath_1^+ +\rho+1}, b)$. Finally, according to (b$_1$) we conclude that for $l = \rho+1, \rho+2, \ldots, m_1-\tau-1$, $v_l^{\rho+1,\tau}$ is proportional to $v_l^{0,\tau}$ on $[x_{\imath_1^+ +\rho+1}, b)$. This completes the proof of (b$'_1$).

The proofs of (b$'_2$)–(b$'_5$) are completely analogous to those of (a$'_2$)–(a$'_5$) in Theorem 4.30, respectively, proceeding now "to the left". This completes the proof of (ii). □

Lemma 4.39. *For* $0 \leqslant \tau < \tau^*$, *the* $(m_1-\tau)$*-dimensional WT-space* $\mathscr{S}_1^{0,\tau}$ *and the basis* $\{V_0^{0,\tau}, \ldots, V_{m_1-\tau-1}^{0,\tau}\}$ *fulfill* (b$_1$)–(b$_5$) *of Theorem* 4.36 *with* $\rho = 0$.

Proof. By Corollary 4.34, the space $\mathscr{S}_1^{0,\tau}$ and the basis $\{V_0^{0,\tau}, \dots, V_{m_1-\tau-1}^{0,\tau}\}$ fulfill the properties (a$_1$)–(a$_5$). Hence, to prove (b$_1$) with $\rho = 0$ it is sufficient to show that for $l = 0, 1, \dots, m_1 - \tau - 1$, $v_l^{0,\tau}$ has no zero interval in $[x_{J_1}, b)$. From Remark 4.35, $v_l^{0,\tau}$ does not go to 0 to the right. Lemma 4.23(a) now implies that for $l = 0, 1, \dots, m_1 - \tau - 1$, $v_l^{0,\tau}$ has no zero interval in $[x_{J_1}, b)$. Thus, (b$_1$) is proved.

Whenever $\rho = 0$, note that the statements (b$_2$), (b$_3$) and (b$_4$) coincide with (a$_2$), (a$_3$) and (a$_4$), respectively. Therefore, Corollary 4.34 implies that (b$_2$), (b$_3$) and (b$_4$) hold.

Finally, to prove (b$_5$) with $\rho = 0$, observe that for $l = 0, 1, \dots, m_1 - \tau - 1$, the spline $v_l^{0,\tau}$ does not go to 0 to the right, since $\tau < \tau^*$. Then (a$_5$) implies (b$_5$) whenever $\rho = 0$. $\square$

Let τ be a fixed and arbitrary integer satisfying $0 \leqslant \tau < \tau^*$. From (a$_1$), $v_0^{0,\tau}$ has no zero interval in $(a, x_{\imath_1})$, whence $v_0^{0,\tau}$ does not go to 0 to the left. We can therefore use Theorem 4.36 to obtain the $(m_1 - \tau - 1)$-dimensional WT-subspace $\mathscr{S}_1^{1,\tau}$ and the basis $\{V_1^{1,\tau}, V_2^{1,\tau}, \dots, V_{m_1-\tau-1}^{1,\tau}\}$, which fulfill properties (b$_1$)–(b$_5$) of Theorem 4.36 with $\rho = 1$. If $1 < n - 1 - \sigma(\imath_1)$ and $v_1^{1,\tau}$ $(= \Psi(V_1^{1,\tau}))$ does not go to 0 to the left, then Theorem 4.36 again applies to obtain the $(m_1 - \tau - 2)$-dimensional WT-subspace $\mathscr{S}_1^{2,\tau}$ and the basis $\{V_2^{2,\tau}, V_3^{2,\tau}, \dots, V_{m_1-\tau-1}^{2,\tau}\}$, which fulfill (b$_1$)–(b$_5$) of Theorem 4.36 with $\rho = 2$. Proceeding in this way, if we define

$$\rho_\tau^* := 1 + \max_{0 \leqslant \rho < n-1-\sigma(\imath_1)} \{\rho : v_\rho^{\rho,\tau} (= \Psi(V_\rho^{\rho,\tau})) \text{ does not go to 0 to the left}\},$$

then we get the WT-subspaces

$$\mathscr{S}_1^{0,\tau} \supset \mathscr{S}_1^{1,\tau} \supset \cdots \supset \mathscr{S}_1^{\rho_\tau^*,\tau},$$

with basis

$$\{V_0^{0,\tau}, \dots, V_{m_1-\tau-1}^{0,\tau}\}, \{V_1^{1,\tau}, \dots, V_{m_1-\tau-1}^{1,\tau}\}, \dots, \{V_{\rho_\tau^*}^{\rho_\tau^*,\tau}, \dots, V_{m_1-\tau-1}^{\rho_\tau^*,\tau}\},$$

respectively. Summarizing, we have the following result.

Corollary 4.40. *For* $0 \leqslant \tau < \tau^*$ *and* $0 \leqslant \rho \leqslant \rho_\tau^*$, *the set* $\mathscr{S}_1^{\rho,\tau}$ *is an* $(m_1 - \tau - \rho)$-*dimensional WT-subspace of* $\mathscr{S}_1^{0,\tau}$. *Moreover, the subspace* $\mathscr{S}_1^{\rho,\tau}$ *and the basis* $\{V_\rho^{\rho,\tau}, V_{\rho+1}^{\rho,\tau}, \dots, V_{m_1-\tau-1}^{\rho,\tau}\}$ *fulfill properties* (b$_1$)–(b$_5$) *of Theorem* 4.36 *as well as Proposition* 4.37 *and Lemma* 4.38.

As with Corollary 4.34, we shall use Corollary 4.40 without an explicit reference whenever we apply (b$_1$)–(b$_5$) of Theorem 4.36, Proposition 4.37 or Lemma 4.38 to the spaces $\mathscr{S}_1^{\rho,\tau}$, $0 \leqslant \tau < \tau^*$; $0 \leqslant \rho \leqslant \rho_\tau^*$.

Lemma 4.41. *For* $\tau = 0, 1, \dots, \tau^* - 1$, *the integer* $\rho_\tau^* (=: \rho^*)$ *does not depend on* τ. *Moreover, for* $\rho = 0, 1, \dots, \rho^*$, *the splines* $v_\rho^{\rho,\tau}$ *and* $v_\rho^{\rho,0}$ *are proportional on* $(a, x_{J_1^- - \tau}]$, *whence they have the same left level.*

Proof. Let τ_0 be an arbitrary integer in $[1, \tau^* - 1]$. From (a_1), v_0^{0,τ_0} and v_0 $(= v_0^{0,0})$ are proportional on $(a, x_{J_1^- - \tau_0}]$, whence they have the same left level. It also follows from (a_1) that both v_0^{0,τ_0} and $v_0^{0,0}$ do not go to 0 to the left. So the spaces $\mathscr{S}_1^{1,\tau_0}$ and $\mathscr{S}_1^{1,0}$ exist, i.e., $\min\{\rho_{\tau_0}^*, \rho_0^*\} \geqslant 1$. For $\rho > 0$ we consider the following inductive hypothesis:

$$\begin{cases} \text{The spaces } \mathscr{S}_1^{\rho,\tau_0} \text{ and } \mathscr{S}_1^{\rho,0} \text{ exist, i.e., } \min\{\rho_{\tau_0}^*, \rho_0^*\} \geqslant \rho; \\ \text{For } \rho' = 0, 1, \dots, \rho - 1, \; v_{\rho'}^{\rho',\tau_0} \text{ and } v_{\rho'}^{\rho',0} \text{ are proportional on } (a, x_{J_1^- - \tau_0}]. \end{cases} \tag{12}$$

We now prove the following claim.

Claim 1. *The spline V_ρ^{ρ,τ_0} is in $\mathscr{S}_1^{\rho,0}$.*

For $\rho' = 0, 1, \dots, \rho - 1$, by construction it follows that V_ρ^{ρ,τ_0} is in $\mathscr{S}_1^{\rho,\tau_0} \subset \mathscr{S}_1^{\rho',\tau_0}$ and also that v_ρ^{ρ,τ_0} has a lower left level than $v_{\rho'}^{\rho',\tau_0}$. So, taking into account the inductive hypothesis, we see that v_ρ^{ρ,τ_0} has also a lower left level than $v_{\rho'}^{\rho',0}$. Thus V_ρ^{ρ,τ_0} is in $\mathscr{S}_1^{\rho,\tau_0} \subset \mathscr{S}_1^{0,\tau_0} \subset \mathscr{S}_1^{0,0}$, and v_ρ^{ρ,τ_0} has a lower left level than $v_{\rho'}^{\rho',0}$ for $\rho' = 0, 1, \dots, \rho - 1$. Hence,

$$V_\rho^{\rho,\tau_0} \in \mathscr{S}_1^{0,0} \cap \mathscr{S}_1^{1,0} \cap \cdots \cap \mathscr{S}_1^{\rho,0} = \mathscr{S}_1^{\rho,0}.$$

This proves the claim.

Note that the restrictions of both V_ρ^{ρ,τ_0} and $V_\rho^{\rho,0}$ to $[x_{\imath_1^+ + \rho}, x_{J_1^- - \tau_0}]$ have $m_1 - \tau_0 - \rho - 1$ simple zeros in this interval, and in addition these zeros are the same for both restrictions. Then using Lemma 4.11(a) it is easy to see that V_ρ^{ρ,τ_0} and $V_\rho^{\rho,0}$ are proportional on $[x_{\imath_1^+ + \rho}, x_{J_1^- - \tau_0}]$, since both restrictions are in the space $\mathscr{S}_{\imath_1^+ + \rho, J_1^- - \tau_0}$. Then there exists a linear combination of both splines, say $V_\rho^{\rho,\tau_0} - \lambda V_\rho^{\rho,0}$ with $\lambda \neq 0$, which vanishes identically on $[x_{\imath_1^+ + \rho}, x_{J_1^- - \tau_0}]$. Moreover, it follows from Claim 1 that $V_\rho^{\rho,\tau_0} - \lambda V_\rho^{\rho,0} \in \mathscr{S}_1^{\rho,0}$. Therefore, Proposition 4.37, with $\tau = 0$ and $i' = \imath_1^+ + \rho$, shows that $v_\rho^{\rho,\tau_0} - \lambda v_\rho^{\rho,0}$ is identically zero in the interval $(a, x_{J_1^- - \tau_0}]$. Hence, v_ρ^{ρ,τ_0} and $v_\rho^{\rho,0}$ are proportional on $(a, x_{J_1^- - \tau_0}]$. So we have two cases to consider. First, both v_ρ^{ρ,τ_0} and $v_\rho^{\rho,0}$ go to 0 to the left. Then $\rho_{\tau_0}^* = \rho_0^* = \rho$ and for $0 \leqslant \rho' \leqslant \rho_{\tau_0}^* = \rho_0^*$, the spline $v_{\rho'}^{\rho',\tau_0}$ is proportional to $v_{\rho'}^{\rho',0}$ on $(a, x_{J_1^- - \tau_0}]$. Secondly, both v_ρ^{ρ,τ_0} and $v_\rho^{\rho,0}$ do not go to 0 to the left. In this case, we again obtain the conditions in (12), now with $\rho + 1$ in the place of ρ, and therefore the same procedure can be applied. Thus, we finally conclude that $\rho_{\tau_0}^* = \rho_0^* = \rho + l$ for some l satisfying $\rho \leqslant \rho + l \leqslant n - 1 - \sigma(\imath_1)$, and that for $\rho' = 0, 1, \dots, \rho_{\tau_0}^* = \rho_0^*$, the splines $v_{\rho'}^{\rho',\tau_0}$ and $v_{\rho'}^{\rho',0}$ are proportional on $(a, x_{J_1^- - \tau_0}]$, whence they have the same left level. This proves the lemma since τ_0 is an arbitrary integer in $[1, \tau^* - 1]$. □

Remark 4.42. Let τ be an integer satisfying $0 \leqslant \tau < \tau^*$. It is clear that $v_\rho^{\rho,\tau}$ does not go to 0 to the left whenever $\rho < \rho^*$. On the contrary, we now assert that $v_{\rho^*}^{\rho^*,\tau}$ $(= \Psi(V_{\rho^*}^{\rho^*,\tau}))$ goes to 0 to the left. Indeed, the assertion follows immediately from the definition of ρ^* $(= \rho_\tau^*)$ provided $\rho^* < n - 1 - \sigma(\imath_1)$. Then assume $\rho^* = n - 1 - \sigma(\imath_1)$. Using Theorem 4.19 we deduce that $\Psi(G)$ goes to 0 to the left for all $G \in \mathscr{S}_{\imath_1,J_1}^+$. Hence, $\mathscr{S}_{\imath_1,J_1}^+ \cap \mathscr{S}_1 \subseteq \mathscr{S}_1^{\rho^*,0}$. Furthermore, $\dim(\mathscr{S}_{\imath_1,J_1}^+ \cap \mathscr{S}_1) = m_1 - (n - 1 - \sigma(\imath_1)) = m_1 - \rho^* = \dim \mathscr{S}_1^{\rho^*,0}$, whence $\mathscr{S}_1^{\rho^*,0} = \mathscr{S}_{\imath_1,J_1}^+ \cap \mathscr{S}_1$ whenever $\rho^* = n - 1 - \sigma(\imath_1)$. Thus, $v_{\rho^*}^{\rho^*,0}$ goes to 0 to the left. Lemma 4.41 now completes the assertion for $v_{\rho^*}^{\rho^*,\tau}$, $0 < \tau < \tau^*$. Finally, using (a) in Lemma 4.38 we conclude that for $l = \rho, \rho + 1, \ldots, m_1 - \tau - 1$, the spline $v_l^{\rho,\tau}$ goes to 0 to the left if and only if $\rho = \rho^*$.

From (b_1), for $0 \leqslant \rho \leqslant \rho^*$ the functions v_ρ^{ρ,τ^*-1} and v_0^{0,τ^*-1} are proportional on $[x_{\imath_1^+ + \rho}, b)$, and therefore they have the same right level. Then using Remark 4.35 we deduce that v_ρ^{ρ,τ^*-1} does not go to 0 to the right. This allows us to define the sets $\mathscr{S}_1^{\rho,\tau^*}$, subspaces of $\mathscr{S}_1^{\rho,\tau^*-1}$, as follows. For $1 \leqslant \rho \leqslant \rho^*$, let

$$\mathscr{S}_1^{\rho,\tau^*} := \{G \in \mathscr{S}_1^{\rho,\tau^*-1}\colon\ \Psi(G) \text{ has a lower right level than } v_\rho^{\rho,\tau^*-1}\}.$$

(Observe that $\mathscr{S}_1^{0,\tau^*}$ was already defined.)

Theorem 4.43. *For* $0 \leqslant \rho \leqslant \rho^*$ *and* $0 \leqslant \tau \leqslant \tau^*$, $\mathscr{S}_1^{\rho,\tau} = \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau}$.

Proof. If $\rho = 0$, then it is obvious that $\mathscr{S}_1^{0,\tau} = \mathscr{S}_1^{0,0} \cap \mathscr{S}_1^{0,\tau}$, since $\mathscr{S}_1^{0,\tau} \subseteq \mathscr{S}_1^{0,0} (= \mathscr{S}_1)$. Thus, assume $\rho > 0$.

We first prove that $\mathscr{S}_1^{\rho,\tau} \subseteq \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau}$ for $0 < \rho \leqslant \rho^*$, $0 \leqslant \tau < \tau^*$. To do this, consider a spline $G \in \mathscr{S}_1^{\rho,\tau}$. Then $G \in \mathscr{S}_1^{\rho-1,\tau}$ and $g := \Psi(G)$ has a lower left level than $v_{\rho-1}^{\rho-1,\tau}$. Hence, Lemma 4.41 implies that g has a lower left level than $v_{\rho-1}^{\rho-1,0}$, and therefore g has a lower left level than $v_{\rho'}^{\rho',0}$ for $\rho' = 0, 1, \ldots, \rho - 1$. So

$$G \in \mathscr{S}_1^{0,0} \cap \mathscr{S}_1^{1,0} \cap \cdots \cap \mathscr{S}_1^{\rho,0} = \mathscr{S}_1^{\rho,0}.$$

On the other hand, by construction, $\mathscr{S}_1^{\rho,\tau} \subset \mathscr{S}_1^{0,\tau}$. Thus, $G \in \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau}$. Let us now show that $\mathscr{S}_1^{\rho,\tau^*} \subseteq \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau^*}$ for $0 < \rho \leqslant \rho^*$. Let $G \in \mathscr{S}_1^{\rho,\tau^*}$. Then $G \in \mathscr{S}_1^{\rho,\tau^*-1}$ and g has a lower right level than v_ρ^{ρ,τ^*-1}. Therefore, from (b_1) applied to V_ρ^{ρ,τ^*-1} we deduce that g has a lower right level than v_ρ^{0,τ^*-1}, and by (a) in Lemma 4.32 it also has a lower right level than v_0^{0,τ^*-1}. Furthermore, applying the previous case we get $G \in \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau^*-1}$. Thus, $G \in \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau^*}$.

We now prove that $\mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau} \subseteq \mathscr{S}_1^{\rho,\tau}$ for $0 < \rho \leqslant \rho^*$ and $0 \leqslant \tau < \tau^*$. Let $G \in \mathscr{S}_1^{\rho,0} \cap \mathscr{S}_1^{0,\tau}$. Then, for $\rho' = 0, 1, \ldots, \rho - 1$, the spline G is in $\mathscr{S}_1^{\rho',0}$ and g has a lower left level than $v_{\rho'}^{\rho',0}$. Hence, Lemma 4.41 implies that g has a lower left level than

$v_{\rho'}^{\rho',\tau}$. So $G \in \mathcal{S}_1^{0,\tau}$ and g has a lower left level that $v_{\rho'}^{\rho',\tau}$ for $\rho' = 0, 1, \ldots, \rho - 1$. Therefore,

$$G \in \mathcal{S}_1^{0,\tau} \cap \mathcal{S}_1^{1,\tau} \cap \cdots \cap \mathcal{S}_1^{\rho,\tau} = \mathcal{S}_1^{\rho,\tau}.$$

Finally, we see that $\mathcal{S}_1^{\rho,0} \cap \mathcal{S}_1^{0,\tau^*} \subseteq \mathcal{S}_1^{\rho,\tau^*}$ for $0 < \rho \leqslant \rho^*$. To this end, consider a $G \in \mathcal{S}_1^{\rho,0} \cap \mathcal{S}_1^{0,\tau^*}$. Then $G \in \mathcal{S}_1^{\rho,0} \cap \mathcal{S}_1^{0,\tau^*-1} = \mathcal{S}_1^{\rho,\tau^*-1}$, where the equality holds because we have just proved it. Moreover, g has a lower right level than v_0^{0,τ^*-1}. From (b$_1$), v_ρ^{0,τ^*-1}, and therefore v_0^{0,τ^*-1} as well, has the same right level as v_ρ^{ρ,τ^*-1}. So $G \in \mathcal{S}_1^{\rho,\tau^*-1}$ and g has a lower right level than v_ρ^{ρ,τ^*-1}. Thus, $G \in \mathcal{S}_1^{\rho,\tau^*}$. This completes the proof of the theorem. □

Corollary 4.44. *The spline* $\Psi(G_1)$ *goes to* 0 *to the right for every* G_1 *in* $\mathcal{S}_1^{\rho,\tau^*}$, $0 \leqslant \rho \leqslant \rho^*$. *The spline* $\Psi(G_2)$ *goes to* 0 *to the left for every* G_2 *in* $\mathcal{S}_1^{\rho^*,\tau}$, $0 \leqslant \tau \leqslant \tau^*$. *Hence, any* G *in* $\mathcal{S}_1^{\rho^*,\tau^*}$ *goes to* 0.

Proof. If G_1 is in $\mathcal{S}_1^{\rho,\tau^*}$, $0 \leqslant \rho \leqslant \rho^*$, then $G_1 \in \mathcal{S}_1^{0,\tau^*}$. On the other hand, we know that v_0^{0,τ^*} goes to 0 to the right. Then using (a) in Lemma 4.32 we deduce that $\Psi(G_1)$ goes to 0 to the right for every G_1 in $\mathcal{S}_1^{\rho,\tau^*} \subset \mathcal{S}_1^{0,\tau^*}$. The proof that $\Psi(G_2)$ goes to 0 to the left for every G_2 in $\mathcal{S}_1^{\rho^*,\tau}$, $0 \leqslant \tau \leqslant \tau^*$, is similar, and so we omit it. The last sentence of the corollary is now obvious. □

Theorem 4.45. *For* $0 \leqslant \rho \leqslant \rho^*$, *the set* $\mathcal{S}_1^{\rho,\tau^*}$ *is an* $(m_1 - \tau^* - \rho)$*-dimensional WT-space.*

Proof. In the case $\rho = 0$ the result follows from Corollary 4.34. Thus, assume $\rho > 0$. From (b$_1$) with $\tau = \tau^* - 1$, for $l = \rho, \rho + 1, \ldots, m_1 - \tau^*$, the spline v_l^{ρ,τ^*-1} is proportional to v_l^{0,τ^*-1} on $[x_{i_1^+ + \rho}, b)$. Hence, v_l^{ρ,τ^*-1} and v_l^{0,τ^*-1} have the same right level. Furthermore, from (a) in Lemma 4.32, v_l^{0,τ^*-1} has the same right level as v_0^{0,τ^*-1}. Then v_l^{ρ,τ^*-1} does not go to 0 to the right because the same fact is valid for v_0^{0,τ^*-1}. Accordingly, the splines $v_\rho^{\rho,\tau^*-1}, v_{\rho+1}^{\rho,\tau^*-1}, \ldots, v_{m_1-\tau^*}^{\rho,\tau^*-1}$ have the same right level, and they do not go to 0 to the right. So we can use a similar procedure to that in the proof of (a$'_1$) in Theorem 4.30. Indeed, as v_ρ^{ρ,τ^*-1} does not go to 0 to the right, Theorem 4.26 shows that there exists $\lim_{x \uparrow b} (g(x)/v_\rho^{\rho,\tau^*-1}(x))$, $x \notin \Omega$, for all $g \in \Psi(\mathcal{S}_1)$. Then it is clear that $\mathcal{S}_1^{\rho,\tau^*}$ is a linear subspace of $\mathcal{S}_1^{\rho,\tau^*-1}$. Let

$$\theta_l^{\rho,\tau^*-1} := \lim_{\substack{x \uparrow b \\ x \notin \Omega}} \frac{v_l^{\rho,\tau^*-1}(x)}{v_\rho^{\rho,\tau^*-1}(x)}, \quad l = \rho + 1, \rho + 2, \ldots, m_1 - \tau^*.$$

As $v_\rho^{\rho,\tau^*-1}, v_{\rho+1}^{\rho,\tau^*-1}, \ldots, v_{m_1-\tau^*}^{\rho,\tau^*-1}$ have the same right level, $0 < |\theta_l^{\rho,\tau^*-1}| < \infty$. Hence, $v_l^{\rho,\tau^*-1} - \theta_l^{\rho,\tau^*-1} v_\rho^{\rho,\tau^*-1}$ has a lower right level than v_ρ^{ρ,τ^*-1}. Moreover, it is easy to see that $\{V_l^{\rho,\tau^*-1} - \theta_l^{\rho,\tau^*-1} V_\rho^{\rho,\tau^*-1}\}_{l=\rho+1}^{m_1-\tau^*}$ is linearly independent. Then $\mathcal{S}_1^{\rho,\tau^*}$ is precisely the $(m_1 - \tau^* - \rho)$-dimensional space spanned by this basis.

We now show that $\mathcal{S}_1^{\rho,\tau^*}$ is a WT-space for $1 \leqslant \rho \leqslant \rho^*$. Let $G \in \mathcal{S}_1^{\rho,\tau^*}$. Then G has at most $m_1 - \tau^* - \rho$ sign changes because $\mathcal{S}_1^{\rho,\tau^*}$ is a subspace of the $(m_1 - \tau^* - \rho + 1)$-dimensional WT-space $\mathcal{S}_1^{\rho,\tau^*-1}$. Suppose first $\rho < \rho^*$. By Remark 4.42, v_ρ^{ρ,τ^*-1} does not go to 0 to the left. So by (b$_5$) the spline v_ρ^{ρ,τ^*-1} satisfies the assumptions on v in Lemma 4.24. On the other hand, it follows from Corollary 4.44 that $g := \Psi(G)$ goes to 0 to the right. If G has $m_1 - \tau^* - \rho$ sign changes, then Lemma 4.24(d) implies that there exists a κ large enough for which $g - \varepsilon v_\rho^{\rho,\tau^*-1}$ has at least $m_\kappa - \tau^* - \rho + 1$ sign changes on $(x_{\imath_\kappa}, x_{\jmath_\kappa})$ for all ε sufficiently small and with a suitable sign. This contradicts (b$_2$) applied to $G - \varepsilon V_\rho^{\rho,\tau^*-1}$, in $\mathcal{S}_1^{\rho,\tau^*-1}$. Therefore, we deduce that G has at most $m_1 - \tau^* - \rho - 1$ sign changes. Thus, $\mathcal{S}_1^{\rho,\tau^*}$ is weak Chebyshev when $\rho < \rho^*$. We finally prove that $\mathcal{S}_1^{\rho^*,\tau^*}$ is weak Chebyshev. Assume to the contrary that $G \in \mathcal{S}_1^{\rho^*,\tau^*}$ has $m_1 - \tau^* - \rho^*$ sign changes. For $\varepsilon_1 \neq 0$, let $G_1 := G - \varepsilon_1 V_{\rho^*}^{\rho^*,\tau^*-1}$ and $g_1 := \Psi(G_1)$. As g goes to 0 and $m_1 - \tau^* - \rho^* > 0$ (so $G \neq 0$), there exists $i > \imath_1^+$ such that $[x_i, b)$ is a maximal zero interval of g. By Proposition 4.37, $i > \imath_1^+ + \rho^*$. Furthermore, by (b$_1$), $v_{\rho^*}^{\rho^*,\tau^*-1}$ is proportional to v_0^{0,τ^*-1} on $[x_{\imath_1^+ + \rho^*}, b)$, and therefore all the zeros of $v_{\rho^*}^{\rho^*,\tau^*-1}$ in $[x_{\imath_1^+ + \rho^*}, b)$ are isolated and of multiplicity 1. We shall use this fact, together with the assumption that G has $m_1 - \tau^* - \rho^*$ sign changes, in the following reasoning. Since g goes to 0 and $v_{\rho^*}^{\rho^*,\tau^*-1}$ goes to 0 to the left, it follows that g_1 goes to 0 to the left, and for an ε_1 sufficiently small and with a suitable sign, g_1 has either at least $m_1 - \tau^* - \rho^* + 1$ sign changes on $(x_{\imath_1}, x_{\jmath_1})$ or at least $m_1 - \tau^* - \rho^*$ sign changes on $(x_{\imath_1}, x_{\jmath_1})$ and a simple zero in $(x_{\jmath_1}, b) \backslash \Omega$, depending on whether $[x_i, b)$ is a maximal zero interval of g with $i \leqslant \jmath_1$, or $i > \jmath_1$, respectively. Note that the first alternative is not possible, since G_1 is in the $(m_1 - \tau^* - \rho^* + 1)$-dimensional WT-space $\mathcal{S}_1^{\rho^*,\tau^*-1}$. As $v_{\rho^*-1}^{\rho^*-1,\tau^*-1}$ satisfies the assumptions on v in Lemma 4.24, apply now (c) in this lemma to conclude that there exists a κ large enough for which $g_1 - \varepsilon v_{\rho^*-1}^{\rho^*-1,\tau^*-1}$ has at least $m_\kappa - \tau^* - \rho^* + 2$ sign changes on $(x_{\imath_\kappa}, x_{\jmath_\kappa})$ for all ε sufficiently small and with a suitable sign. This contradicts (b$_2$) applied to $G_1 - \varepsilon V_{\rho^*-1}^{\rho^*-1,\tau^*-1}$, in $\mathcal{S}_1^{\rho^*-1,\tau^*-1}$. Thus, $\mathcal{S}_1^{\rho^*,\tau^*}$ is weak Chebyshev. $\square$

Theorem 4.46. *Let $G \in \mathcal{S}_1$ with the property that $g := \Psi(G)$ has the same right level as $v_0^{0,\tau'}$ for some $0 \leqslant \tau' \leqslant \tau^*$ and the same left level as $v_{\rho'}^{\rho',0}$ for some $0 \leqslant \rho' \leqslant \rho^*$. If H is in $\mathcal{S}_1^{\rho_0,\tau_0}$, being $\rho_0 \geqslant \rho'$ and $\tau_0 \geqslant \tau'$, and g is in $\mathcal{L}_\phi$, then also $h := \Psi(H)$ is in $\mathcal{L}_\phi$.*

Proof. As $H \in \mathcal{S}_1^{\rho_0,\tau_0}$, it follows from Theorem 4.43 that $H \in \mathcal{S}_1^{\rho_0,0} \cap \mathcal{S}_1^{0,\tau_0}$. Note that $\mathcal{S}_1^{\rho_0,0} \subseteq \mathcal{S}_1^{\rho',0}$ and $\mathcal{S}_1^{0,\tau_0} \subseteq \mathcal{S}_1^{0,\tau'}$, since $\rho_0 \geqslant \rho'$ and $\tau_0 \geqslant \tau'$. Then taking into account that

g has the same right level as $v_0^{0,\tau'}$ and the same left level as $v_{\rho'}^{\rho',0}$, and using (a) in Lemmas 4.32 and 4.38, we see that g does not have a lower right or left level than h. Hence, it is easy to show that $|h| \leqslant \lambda |g|$ on $J \backslash K_0$ for some constant λ and some compact interval $K_0 \subset J$. Therefore, $\int_{J \backslash K_0} \phi(|h|) \leqslant \int_{J \backslash K_0} \phi(\lambda |g|)$, since ϕ is increasing. As $g \in \mathscr{L}_\phi$, and $\mathscr{L}_\phi$ is a linear space, $\int_J \phi(\lambda|g|) < \infty$. On the other hand, it is clear that $\int_{K_0} \phi(|h|) < \infty$. Thus, $\int_J \phi(|h|) < \infty$, which proves the theorem. □

Lemma 4.47. *If $h \in \mathscr{S}_\Pi \backslash \{0\}$ satisfies $h(z) = 0$ for all $z \in \Omega$, then h has at most one maximal zero interval. Hence, there exists no $H \in \mathscr{S}_1^{\rho^*,\tau^*} \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set, whence*

$$m_1 - \tau^* - \rho^* - 1 < \Omega_{(\imath_1, J_1)}.$$

Proof. Assume that there exists an $h \in \mathscr{S}_\Pi \backslash \{0\}$ satisfying $h(z) = 0$ for all $z \in \Omega$, and with at least two maximal zero intervals. Then there exist indexes $i_0, j_0 \in \mathbb{Z}$, $i_0 < j_0$, such that the restriction of h to $[x_{i_0}, x_{j_0}]$ is in $\mathscr{S}^0_{i_0,j_0}$ without zero intervals. Thus,

$$\Omega_{(i_0,j_0)} \leqslant Z_{(i_0,j_0)}(h) \leqslant j_0 - i_0 - n,$$

where the last inequality is due to Lemma 4.11(c). This contradicts that $\{u, \Omega\}$ is a reference pair (Definition 4.18), which proves the first sentence of the lemma.

To prove the second statement in the lemma, suppose that there exists $H \in \mathscr{S}_1^{\rho^*,\tau^*} \backslash \{0\}$ such that $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set for H. Then $h := \Psi(H)$ has at least two maximal zero intervals, since h goes to 0 (Corollary 4.44). Moreover, as $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set for H, using Theorem 4.19 we see that $h(z) = 0$ for all $z \in \Omega$. This contradicts the stated above. In particular, this is the case if $m_1 - \tau^* - \rho^* - 1 \geqslant \Omega_{(\imath_1, J_1)}$. Indeed, after applying [SSS] if $m_1 - \tau^* - \rho^* - 1 > \Omega_{(\imath_1, J_1)}$, we can use [JKZ] in an $(\Omega_{(\imath_1, J_1)} + 1)$-dimensional WT-subspace of $\mathscr{S}_1^{\rho^*,\tau^*}$ to obtain an $H_0 \in \mathscr{S}_1^{\rho^*,\tau^*} \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set. □

Now we shall deal with the spline u itself. Let U be the restriction of u to $[x_{\imath_1}, x_{J_1}]$. By (3), $U \in \mathscr{S}_1$, and from (3) and Theorem 4.19, $u = \Psi(U)$.

Lemma 4.48. *There exist integers $\tau(u)$ and $\rho(u)$, $0 \leqslant \tau(u) < \tau^*$, $0 \leqslant \rho(u) < \rho^*$, such that u has the same right level as $v_0^{0,\tau(u)}$, and the same left level as $v_{\rho(u)}^{\rho(u),0}$. Moreover,*

$$\Omega_{(\imath_1, J_1)} \leqslant m_1 - \tau(u) - \rho(u) - 1.$$

Proof. As $\{u, \Omega\}$ is a reference pair, u does not go to 0 to the right, and therefore we see that there exists $\tau(u) \in \mathbb{Z}$ satisfying $0 \leqslant \tau(u) < \tau^*$, and such that u has the same right level as $v_0^{0,\tau(u)}$. Likewise, as u does not go to 0 to the left, there exists $\rho(u)\in\mathbb{Z}$, satisfying $0\leqslant\rho(u)<\rho^*$, and such that u has the same left level as $v_{\rho(u)}^{\rho(u),0}$.

By (2), $\imath_1\leqslant -n<n\leqslant \jmath_1$. So it is easy to see that there exists a unique spline $G_1\in\mathscr{S}^0_{\imath_1,\jmath_1}$ such that $G_1=1$ on $[x_{\imath_1+n-1},x_{\jmath_1-n+1}]$. Furthermore, $G_1>0$ on $(x_{\imath_1},x_{\jmath_1})$. Then the following facts can be easily proved:

(i) For all ε sufficiently small, the r simple zeros of U in $(x_{\imath_1},x_{\jmath_1})$ produce r simple zeros of $U_\varepsilon := U-\varepsilon G_1$ in $(x_{\imath_1},x_{\jmath_1})$.
(ii) For all ε sufficiently small and with a suitable sign, the t double zeros of U in $(x_{\imath_1},x_{\jmath_1})$ produce at least t simple zeros of U_ε in $(x_{\imath_1},x_{\jmath_1})$.

Moreover, taking ε sufficiently small, all the simple zeros of U_ε obtained in (i) and (ii) are different. Thus, we can choose an $\varepsilon\neq 0$, small enough and with a suitable sign, so that U_ε has at least $Z_{(\imath_1,\jmath_1)}(U)$ sign changes on $(x_{\imath_1},x_{\jmath_1})$. Then the number of sign changes of U_ε is not smaller than $\Omega_{(\imath_1,\jmath_1)}$, since by definition of Ω, $Z_{(\imath_1,\jmath_1)}(U)\geqslant\Omega_{(\imath_1,\jmath_1)}$. On the other hand, $u_\varepsilon := \Psi(U_\varepsilon)$ has the same level as u because $u_\varepsilon = u$ on $J\backslash(x_{\imath_1},x_{\jmath_1})$. Hence, u_ε has the same right level as $v_0^{0,\tau(u)}$ and the same left level as $v_{\rho(u)}^{\rho(u),0}$. Then we deduce that U_ε is in $\mathscr{S}_1^{\rho(u),0}\cap\mathscr{S}_1^{0,\tau(u)}=\mathscr{S}_1^{\rho(u),\tau(u)}$, where the equality is due to Theorem 4.43. In this way, as $\mathscr{S}_1^{\rho(u),\tau(u)}$ is an $(m_1-\rho(u)-\tau(u))$-dimensional WT-space (Corollary 4.40), and the number of sign changes of U_ε is not smaller than $\Omega_{(\imath_1,\jmath_1)}$, we conclude that

$$\Omega_{(\imath_1,\jmath_1)}\leqslant m_1-\tau(u)-\rho(u)-1. \qquad \square$$

According to Lemmas 4.47 and 4.48,

$$m_1-\tau^*-\rho^*-1<\Omega_{(\imath_1,\jmath_1)}\leqslant m_1-\tau(u)-\rho(u)-1.$$

From now on, we shall denote by ρ_0 and τ_0 any pair of integers satisfying

$$\begin{cases}\rho(u)\leqslant\rho_0\leqslant\rho^* \quad \text{and} \quad \tau(u)\leqslant\tau_0\leqslant\tau^*;\\ m_1-\tau_0-\rho_0-1=\Omega_{(\imath_1,\jmath_1)}.\end{cases} \tag{13}$$

Note that $\rho_0+\tau_0<\rho^*+\tau^*$, since $\Omega_{(\imath_1,\jmath_1)}>m_1-\tau^*-\rho^*-1$.

Lemma 4.49. *Let ρ_0 and τ_0 be integers satisfying the conditions in* (13). *To prove Theorem* 4.12 *whenever $\{u,\Omega\}$ is a reference pair, it is sufficient to show that there exists an $H_0\in\mathscr{S}_1^{\rho_0,\tau_0}\backslash\{0\}$ such that Ω is an alternating set for $\Psi(H_0)$.*

Proof. Recall that $\Omega=\{z\in J\colon z \text{ is a simple zero of } u^*\}$, where u^* is a continuous function satisfying $|u^*|=|u|$. Assume that Ω is an alternating set for $h_0:=\Psi(H_0)$, where $H_0\in\mathscr{S}_1^{\rho_0,\tau_0}\backslash\{0\}$. Then it is clear that either h_0 or $-h_0$ satisfies (a) and (b) of Theorem 4.12. Let U be the restriction of u to $[x_{\imath_1},x_{\jmath_1}]$. As $\rho_0\geqslant\rho(u)$ and $\tau_0\geqslant\tau(u)$, applying Theorem 4.46, with U in place of G and H_0 in place of H, we see that (c) of Theorem 4.12 also holds. □

By (13), $m_1 - \tau_0 - \rho_0 - 1 = \Omega_{(\imath_1, J_1)}$. Then we can use Property [JKZ] in the $(m_1 - \tau_0 - \rho_0)$-dimensional WT-space $\mathscr{S}_1^{\rho_0,\tau_0}$ to obtain an $H \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set. So, in view of Lemma 4.49, the problem is now to prove that Ω is an alternating set for $\Psi(H)$. This will be immediately achieved when H has no zero interval.

Lemma 4.50. *Let ρ_0 and τ_0 be integers satisfying the conditions in* (13). *If $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set for H_0, in $\mathscr{S}_1^{\rho_0,\tau_0}$ and without zero interval, then Ω is an alternating set for $h_0 := \Psi(H_0)$.*

Proof. Let $\Omega \cap (x_{\imath_1}, x_{J_1})$ be an alternating set for $H_0 \in \mathscr{S}_1^{\rho_0,\tau_0}$, and assume that H_0 has no zero interval. By (13), $m_1 - \tau_0 - \rho_0 - 1 = \Omega_{(\imath_1, J_1)}$. Therefore, H_0 has $m_1 - \tau_0 - \rho_0 - 1$ sign changes. Suppose that h_0 does not go to 0 to the right nor to the left. Then, in particular, $\tau_0 < \tau^*$, and so (b$_4$) applied to $H_0 \in \mathscr{S}_1^{\rho_0,\tau_0}$ implies that h_0 has no double zero, and

$$\{h_0 = 0\} \cap (J \backslash (x_{\imath_1}, x_{J_1})) = \Omega \cap (J \backslash (x_{\imath_1}, x_{J_1})).$$

Thus, Ω is an alternating set for h_0, and therefore the lemma holds in this case.

Assume now that h_0 goes to 0 to the right. Then h_0 has a maximal zero interval, say $[x_j, b)$. By hypothesis, $j \geqslant J_1$. From Lemma 4.47, h_0 does not go to 0 to the left, and by Corollary 4.44, $\rho_0 < \rho^*$. Let $\tau_1 = \max\{0, \tau_0 - 1\}$. Then $V_0^{\rho_0,\tau_1}$ is in $\mathscr{S}_1^{\rho_0,\tau_1}$ and so it satisfies the hypothesis on V in Lemma 4.24. If h_0 has a double zero in (a, x_j), or a simple zero in $(a, x_j) \backslash \Omega$ (note that h_0 cannot have simple zeros in $(x_{\imath_1}, x_{J_1}) \backslash \Omega$), then applying Lemma 4.24 we deduce that there exists a κ sufficiently large for which $h_0 - \varepsilon v_0^{\rho_0,\tau_1}$ has at least $m_\kappa - \tau_0 - \rho_0 + 1$ sign changes on $(x_{\imath_\kappa}, x_{J_\kappa})$ for all ε sufficiently small and with a suitable sign. This contradicts (b$_2$) applied to $H_0 - \varepsilon V_0^{\rho_0,\tau_1}$, in $\mathscr{S}_1^{\rho_0,\tau_1}$. An analogous argument proceeds when h_0 goes to 0 only to the left. So we conclude that in every case Ω is an alternating set for h_0. □

Remark 4.51. Let ρ_0 and τ_0 be integers satisfying the conditions in (13). Then applying Property [JKZ] in the $(m_1 - \tau_0 - \rho_0)$-dimensional WT-space $\mathscr{S}_1^{\rho_0,\tau_0}$ we obtain a spline $H \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set. If H has no zero interval, then it follows from Lemmas 4.49 and 4.50, with $H_0 := H$, that Theorem 4.12 is true when $\{u, \Omega\}$ is a reference pair. Therefore, to complete the proof of Theorem 4.12 whenever $\{u, \Omega\}$ is a reference pair, it remains to consider the case in which the spline H has a zero interval. Note that in this situation it is not possible to assure that Ω is an alternating set for $\Psi(H)$. So, in order to apply Lemma 4.49 in this case, we shall need further results. The existence of H allows us to observe that there exists an $\bar{H} \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{J_1})$ is an alternating set, and such that either $\bar{H}$ coincides with H on $[x_{\ell_0}, x_{J_1}]$, and $[x_{\imath_1}, x_{\ell_0}]$ is a maximal zero interval of $\bar{H}$, $\imath_1 < \ell_0 < \jmath_1^-$, or $\bar{H}$ coincides with H on $[x_{\imath_1}, x_{\ell_1}]$, and $[x_{\ell_1}, x_{\jmath_1}]$ is a maximal zero interval of $\bar{H}$, $\imath_1^+ < \ell_1 < \jmath_1$. Without loss of generality, we shall treat the first case. The other is completely symmetrical on its conditions, and therefore on its treatment as well.

Definition 4.52. For each $\ell \in \mathbb{Z} \cap (\imath_1, \jmath_1)$, and for every $\tau = 0, 1, \dots, \tau^*$, we define

$$\mathscr{T}_\ell^\tau := \{G \in \mathscr{S}_1^{0,\tau} : [x_{\imath_1}, x_\ell] \text{ is a zero interval of } G\}.$$

It is clear that $\mathscr{T}_\ell^\tau$ is a linear subspace of $\mathscr{S}_1^{\rho^*,\tau}$. We now claim that if $\jmath_1^- - \ell - \tau \leqslant 0$, then $\mathscr{T}_\ell^\tau = \{0\}$. Indeed, this follows immediately if $\tau = 0$ and $\ell = \jmath_1^-$. If $\tau > 0$ then $\mathscr{T}_\ell^\tau \subset \mathscr{S}_1^{0,\tau} \subset \mathscr{S}_1^{0,\tau-1}$, and (c) in Lemma 4.32 applied to $\mathscr{S}_1^{0,\tau-1}$ shows that every G in $\mathscr{S}_1^{0,\tau-1} \backslash \{0\}$ vanishing on $[x_{\imath_1}, x_{\jmath_1^- - \tau}]$ is such that $\Psi(G)$ has the same right level as $v_0^{0,\tau-1}$. Hence, G cannot be in $\mathscr{S}_1^{0,\tau} \supset \mathscr{T}_\ell^\tau$, which proves the claim. In Theorem 4.55, we shall prove that $\mathscr{T}_\ell^\tau$ is a $(\jmath_1^- - \ell - \tau)$-dimensional WT-space whenever $\jmath_1^- - \ell - \tau > 0$.

Remark 4.53. Take points $\bar{y}_1 < \bar{y}_2 < \cdots < \bar{y}_{m_1-1}$ in $(x_{\imath_1}, x_{\jmath_1^-})$ in such a way that $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{n-1-\sigma(\imath_1)}$ are in $(x_{\imath_1}, x_{\imath_1+1})$, and $\bar{y}_k$ is in $(x_{k-n+\imath_1^++1}, x_{k-n+\imath_1^++2})$, $k = n - \sigma(\imath_1), \dots, m_1 - 1$. For $\tau = 0, 1, \dots, \tau^* - 1$, consider the set $\{\bar{y}_k\}_{k=1}^{m_1-\tau-1}$ in $(x_{\imath_1}, x_{\jmath_1^- - \tau})$. Using Property [JKZ] in the $(m_1 - \tau)$-dimensional WT-space $\mathscr{S}_1^{0,\tau}$ we obtain $Q^\tau \in \mathscr{S}_1^{0,\tau} \backslash \{0\}$ for which $\{\bar{y}_k\}_{k=1}^{m_1-\tau-1}$ is an alternating set. If Q^τ has a zero interval in $[x_{\imath_1}, x_{\jmath_1^- - \tau}]$, then the location of the points $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{m_1-\tau-1}$ implies that $[x_{\imath_1}, x_{\jmath_1^- - \tau}]$ has to be a zero interval of Q^τ, but this is not possible because of Proposition 4.31. Then for $\tau = 0, 1, \dots, \tau^* - 1$, Q^τ has no zero interval in $[x_{\imath_1}, x_{\jmath_1^- - \tau}]$, whence it has $m_1 - \tau - 1$ sign changes on $(x_{\imath_1}, x_{\jmath_1})$. Then from (b) in Lemma 4.32 applied to $Q^\tau \in \mathscr{S}_1^{0,\tau}$, q^τ has the same right level as $v_0^{0,\tau}$, and therefore q^τ does not go to 0 to the right. Analogously, from (b) in Lemma 4.38 applied to $Q^\tau \in \mathscr{S}_1^{0,\tau}$, q^τ has the same left level as $v_0^{0,\tau}$, and therefore q^τ does not go to 0 to the left. As a consequence of all these results, from Proposition 4.31 and Lemma 4.23 it is easy to see that q^τ has no zero interval. Furthermore, (a_4) of Theorem 4.30 applied to $Q^\tau \in \mathscr{S}_1^{0,\tau}$ implies that q^τ has no double zero and

$$\{q^\tau = 0\} \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})) = \Omega \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})).$$

Thus, q^τ has $m_\kappa - \tau - 1$ simple zeros in $(x_{\imath_\kappa}, x_{\jmath_\kappa})$ for every $\kappa \in \mathbb{N}$. We now assert that all the zeros of q^τ have multiplicity one. In the case $\tau = 0$, the assertion follows from Lemma 3.21 applied to Q^0. Assuming that the assertion holds for $\tau - 1$, being $\tau \geqslant 1$, we will prove it for τ. Note that by construction, the zeros of Q^τ are also zeros of $Q^{\tau-1}$, and moreover, the zeros of q^τ are zeros of $q^{\tau-1}$, since

$$\{q^\tau = 0\} \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})) = \{q^{\tau-1} = 0\} \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})) = \Omega \cap (J \backslash (x_{\imath_1}, x_{\jmath_1})).$$

Thus, if q^τ has a zero of odd multiplicity greater than 2 at a point z, then it is easy to see that for all ε sufficiently small and with a suitable sign, $q^\tau - \varepsilon q^{\tau-1}$ has at least three simple zeros, say $z_1(\varepsilon) = z$, $z_2(\varepsilon)$ and $z_3(\varepsilon)$, with $z_2(\varepsilon)\uparrow z$ and $z_3(\varepsilon)\downarrow z$ as $\varepsilon\to 0$. Let $\kappa\in\mathbb{Z}$ such that $z\in(x_{\imath_\kappa}, x_{\jmath_\kappa})$. Then $q^\tau - \varepsilon q^{\tau-1}$ would have at least $m_\kappa - \tau + 1$ simple zeros in $(x_{\imath_\kappa}, x_{\jmath_\kappa})$ for all ε sufficiently small and with a suitable sign, which contradicts (a$_2$) applied to $Q^\tau - \varepsilon Q^{\tau-1}$, in $\mathscr{S}_1^{0,\tau-1}$. Thus, the assertion is proved. Summarizing, for $\tau = 0, 1, \dots, \tau^* - 1$, we have proved that there exists a $Q^\tau\in\mathscr{S}_1^{0,\tau}$ such that the following properties hold:

- Q^τ has $m_1 - \tau - 1$ sign changes on $(x_{\imath_1}, x_{\jmath_1^- - \tau})$
- q^τ has the same right level as $v_0^{0,\tau}$
- all the zeros of q^τ are isolated and of multiplicity 1, and

$$\{q^\tau = 0\}\cap(J\backslash(x_{\imath_1}, x_{\jmath_1})) = \Omega\cap(J\backslash(x_{\imath_1}, x_{\jmath_1})).$$

Lemma 4.54. *Assume that* G_0 *is in* $\mathscr{T}_\ell^\tau\backslash\{0\}$ *with* $\jmath_1^- - \ell - \tau - 1$ *sign changes,* $\imath_1 < \ell < \jmath_1$, $0\leqslant\tau\leqslant\tau^* - 1$. *Then the following properties hold*:

(a) $g_0 := \Psi(G_0)$ *has neither double zeros nor zero intervals in* (x_ℓ, b), *and*

$$\{g_0 = 0\}\cap[x_{\jmath_1}, b) = \Omega\cap[x_{\jmath_1}, b).$$

(b) g_0 *has the same right level as* $v_0^{0,\tau}$.

Proof. Let G_0 be as in the statement of the lemma. Observe that $\jmath_1^- - \ell - \tau > 0$, since $G_0\in\mathscr{T}_\ell^\tau\backslash\{0\}$. Note also that $[x_{\imath_1}, x_\ell]$ is a zero interval of G_0. Suppose that $[x_{\imath_1}, x_j]$ is a zero interval of G_0, $\ell\leqslant j\leqslant\ell + 1$. By construction, Q^τ has $n - 1 - \sigma(\imath_1) + j - \imath_1 - 1 = j - \imath_1^+ + n - 2$ sign changes on $(x_{\imath_1}, x_j)$. Then since G_0 has $\jmath_1^- - \ell - \tau - 1$ sign changes and $[x_{\imath_1}, x_j]$ is a zero interval of G_0, the spline $G_0 - \varepsilon Q^\tau$ has at least $(j - \imath_1^+ + n - 2) + (\jmath_1^- - \ell - \tau - 1) + 1 = m_1 - \tau - 1 + j - \ell$ sign changes for all ε sufficiently small and with a suitable sign. Then we deduce that $j = \ell$ necessarily. Otherwise (a$_2$) applied to $G_0 - \varepsilon Q^\tau\in\mathscr{S}_1^{0,\tau}$ is contradicted. Thus, $[x_{\imath_1}, x_\ell]$ is a maximal zero interval of G_0.

Now, with similar proofs to those of Claims 2 and 3 in Lemma 4.32(c), we below deduce Claims 1 and 2, respectively. We only remark that $v_0^{0,\tau}$, used in Lemma 4.32, must be here replaced by q^τ.

Claim 1. *For all* ε *sufficiently small*, $g_0 - \varepsilon q^\tau$ *has no double zero*, *and*

$$\{g_0 - \varepsilon q^\tau = 0\}\cap(J\backslash(x_{\imath_1}, x_{\jmath_1})) = \Omega\cap(J\backslash(x_{\imath_1}, x_{\jmath_1})).$$

Claim 2. *The spline g_0 has no zero interval in $[x_\ell, b)$.*

Taking into account Claim 2, to prove (a) it remains to show that g_0 has no double zero in (x_ℓ, b), and that g_0 has no simple zero in $[x_{J_1}, b)\backslash\Omega$. Assume first that g_0 has a double zero in (x_ℓ, x_{J_1}). Then it is easy to see that we can choose $\varepsilon \neq 0$, small enough and with a suitable sign, so that $g_0 - \varepsilon q^\tau$ has

$$(\ell - \imath_1^+ + n - 2) + (\jmath_1^- - \tau - \ell - 1) + 2 = m_1 - \tau$$

sign changes on $(x_{\imath_1}, x_{J_1})$. This is a contradiction, because $G_0 - \varepsilon Q^\tau$ is in the $(m_1 - \tau)$-dimensional WT-space $\mathscr{S}_1^{0,\tau}$. Thus, g_0 has no double zero in (x_ℓ, x_{J_1}). Suppose now that g_0 has a double zero in $[x_{J_1}, b)$. Then it is not difficult to see that for all $\varepsilon \neq 0$, small enough and with a suitable sign, $g_0 - \varepsilon q^\tau$ has at least a simple zero in $(x_{J_1}, b)\backslash\Omega$. This contradicts Claim 1. In consequence, g_0 has no double zero in (x_ℓ, b). Finally, if g_0 has a simple zero in $[x_{J_1}, b)\backslash\Omega$, then $g_0 - \varepsilon q^\tau$ has a simple zero in $[x_{J_1}, b)\backslash\Omega$ for all ε sufficiently small, and Claim 1 is again contradicted. This completes the proof of (a).

We now prove (b). We have shown that g_0, q^τ (see Remark 4.53), and $g_0 - \varepsilon q^\tau$ for all ε sufficiently small, have no double zero and

$$\{q^\tau = 0\} \cap [x_{J_1}, b) = \{g_0 - \varepsilon q^\tau = 0\} \cap [x_{J_1}, b)$$
$$= \{g_0 = 0\} \cap [x_{J_1}, b) = \Omega \cap [x_{J_1}, b).$$

Accordingly, Lemma 4.28(a) shows that g_0 does not have a lower right level than q^τ. Then we deduce that g_0 does not have a lower right level than $v_0^{0,\tau}$, because q^τ has the same right level as $v_0^{0,\tau}$ (Remark 4.53). Finally, using (a) in Lemma 4.32, we conclude that g_0 has the same right level as $v_0^{0,\tau}$. $\square$

Theorem 4.55. *For every $\tau = 0, 1, \ldots, \tau^*$, the set $\mathscr{T}_\ell^\tau$ is a $(\jmath_1^- - \ell - \tau)$-dimensional WT-space whenever $\jmath_1^- - \ell - \tau > 0$.*

Proof. The proof is by induction on τ. Observe first that the space $\mathscr{T}_\ell^0$ can be identified with $\mathscr{S}_{\ell,\jmath_1^-}^+$, and therefore $\mathscr{T}_\ell^0$ is indeed a $(\jmath_1^- - \ell)$-dimensional WT-space whenever $\jmath_1^- - \ell > 0$. Thus, assume $0 \leqslant \tau < \min\{\tau^*, \jmath_1^- - \ell - 1\}$ and suppose that $\mathscr{T}_\ell^\tau$ is a $(\jmath_1^- - \ell - \tau)$-dimensional WT-space.

In Remark 4.53 we took points $\bar{y}_1 < \bar{y}_2 < \cdots < \bar{y}_{m_1 - 1}$ in $(x_{\imath_1}, x_{\jmath_1^-})$ in such a way that $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n-1-\sigma(\imath_1)}$ are in the component $(x_{\imath_1}, x_{\imath_1+1})$, and $\bar{y}_k$ is in the component $(x_{k-n+\imath_1^++1}, x_{k-n+\imath_1^++2})$, $k = n - \sigma(\imath_1), \ldots, m_1 - 1$. So for each $\tau = 1, 2, \ldots, \tau^*$, the set $\{\bar{y}_k\}_{m_1-(\jmath_1^- - \ell - 1)}^{m_1-\tau-1}$ is contained in $(x_{\ell+1}, x_{\jmath_1^- - \tau})$. For $k = m_1 - (\jmath_1^- - \ell - 1), \ldots, m_1 - \tau - 1$, we now consider points $\bar{y}_k'$ so that $\bar{y}_k' \neq \bar{y}_k$ and $\bar{y}_k'$ is in the same component in which $\bar{y}_k$ is. Now, for $l = m_1 - (\jmath_1^- - \ell), \ldots, m_1 - \tau - 1$, let $\bar{y}_k^l := \bar{y}_k$ if $k \neq l$, and $\bar{y}_l^l := \bar{y}_l'$. Using Property [JKZ] in the WT-space $\mathscr{T}_\ell^\tau$ we obtain W_l^τ in $\mathscr{T}_\ell^\tau\backslash\{0\}$ for which $\{\bar{y}_k^l\}_{k=m_1-(\jmath_1^- - \ell - 1)}^{m_1-\tau-1}$ is an alternating set. We assert that W_l^τ has no zero interval in $[x_\ell, x_{J_1^- -\tau}]$. Assume to the contrary that for some l', $W_{l'}^\tau$ has a zero interval in $[x_\ell, x_{J_1^- -\tau}]$. Then the location of the points $\bar{y}_k^{l'}$ implies that $[x_\ell, x_{J_1^- -\tau}]$ is a zero interval of $W_{l'}^\tau$, and therefore $W_{l'}^\tau$ vanishes identically on $[x_{\imath_1}, x_{J_1^- -\tau}]$. As $W_{l'}^\tau$ is in $\mathscr{T}_\ell^\tau \subset \mathscr{S}_1^{0,\tau}$, Proposition 4.31 applied to $W_{l'}^\tau \in \mathscr{S}_1^{0,\tau}$ shows that $W_{l'}^\tau$ vanishes identically on $[x_{\imath_1}, x_{J_1}]$, which is a contradiction. This proves the assertion. Then for every l, W_l^τ changes sign at each $\bar{y}_k^l$, $k = m_1 - (J_1^- - \ell - 1), \dots, m_1 - \tau - 1$. As the restriction of W_l^τ to $[x_\ell, x_{J_1^- -\tau}]$ is in $\mathscr{S}^+_{\ell, J_1^- -\tau}$, it follows from Lemma 4.11(b) that the $J_1^- - \tau - \ell - 1$ points in $\{\bar{y}_k^l\}_{k=m_1-(J_1^- -\ell-1)}^{m_1-\tau-1}$ are the only zeros (of multiplicity 1) of W_l^τ in $(x_\ell, x_{J_1^- -\tau}]$. Therefore, the replacement method employed in the construction of $\{W_l^\tau\}_{l=m_1-(J_1^- -\ell)}^{m_1-\tau-1}$ insures that this set is linearly independent in $\mathscr{T}_\ell^\tau$, and thus $\{W_l^\tau\}_{l=m_1-(J_1^- -\ell)}^{m_1-\tau-1}$ is a basis for the $(J_1^- - \ell - \tau)$-dimensional WT-space $\mathscr{T}_\ell^\tau$. For $l = m_1 - (J_1^- - \ell), \dots, m_1 - \tau - 1$, applying now Lemma 4.54(b) to W_l^τ, in $\mathscr{T}_\ell^\tau \backslash \{0\}$ and with $J_1^- - \ell - \tau - 1$ sign changes, we see that $w_l^\tau := \Psi(W_l^\tau)$ has the same right level as $v_0^{0,\tau}$. Then w_l^τ does not go to 0 to the right, since $\tau < \tau^*$. Accordingly, there exists

$$\theta_l^\tau := \lim_{\substack{x \uparrow b \\ x \notin \Omega}} \frac{w_l^\tau(x)}{w_{m_1-\tau-1}^\tau(x)}, \quad l = m_1 - (J_1^- - \ell), \dots, m_1 - \tau - 2,$$

and $0 < |\theta_l^\tau| < \infty$. Hence, each $W_l^\tau - \theta_l^\tau W_{m_1-\tau-1}^\tau$ is in $\mathscr{T}_\ell^{\tau+1}$. Moreover, it is easily seen that

$$\{W_l^\tau - \theta_l^\tau W_{m_1-\tau-1}^\tau\}_{l=m_1-(J_1^- -\ell)}^{m_1-\tau-2}$$

is linearly independent. Consequently, $\mathscr{T}_\ell^{\tau+1}$ is precisely the $(J_1^- - \ell - \tau - 1)$-dimensional space spanned by this basis. We now prove that $\mathscr{T}_\ell^{\tau+1}$ is weak Chebyshev. To do this, let $G \in \mathscr{T}_\ell^{\tau+1} \subset \mathscr{T}_\ell^\tau$. By the inductive hypothesis, $\mathscr{T}_\ell^\tau$ is a $(J_1^- - \ell - \tau)$-dimensional WT-space. Then G has at most $J_1^- - \ell - \tau - 1$ sign changes. On the other hand, $\Psi(G)$ has a lower right level than $v_0^{0,\tau}$. Then from Lemma 4.54(b), G has at most $J_1^- - \ell - \tau - 2$ sign changes. So $\mathscr{T}_\ell^{\tau+1}$ is weak Chebyshev. □

Lemma 4.56. *Assume that G is in $\mathscr{T}_\ell^\tau$ without zero intervals in $[x_\ell, x_{J_1}]$, $\imath_1 < \ell < J_1$, $0 \leqslant \tau \leqslant \tau^* - 1$. Then*

$$Z_{(\ell, J_1)}(G) \leqslant J_1^- - \ell - \tau - 1.$$

Proof. Let G be as in the statement of the lemma, and consider the spline $Q^\tau \in \mathscr{T}_\ell^\tau \backslash \{0\}$ introduced in Remark 4.53. As all the zeros of Q^τ have multiplicity one and Q^τ has $\ell - \imath_1^+ + n - 2$ simple zeros in $(x_{\imath_1}, x_\ell)$, the following facts can be easily checked:

(i) For all ε, the spline $G - \varepsilon Q^\tau$ has $\ell - \imath_1^+ + n - 2$ simple zeros in $(x_{\imath_1}, x_\ell)$.

(ii) For all ε sufficiently small and with a suitable sign, $G - \varepsilon Q^\tau$ has a simple zero $z(\varepsilon) \in (x_\ell, x_{\ell+1})$, being $z(\varepsilon) \downarrow x_\ell$ as $\varepsilon \to 0$.
(iii) For all ε sufficiently small, each simple zero of G produces a simple zero of $G - \varepsilon Q^\tau$.
(iv) The t double zeros of G produce either t simple zeros of $G - \varepsilon Q^\tau$ for all ε sufficiently small or at least $t + 1$ simple zeros of $G - \varepsilon Q^\tau$ for all ε sufficiently small and with a suitable sign.

Moreover, it is clear that for ε sufficiently small, all the zeros of $G - \varepsilon Q^\tau$ obtained in (i)–(iv) are different. Accordingly, it is easy to check that we can take $\varepsilon \neq 0$, small enough and with a suitable sign, so that $G - \varepsilon Q^\tau$ has at least

$$\ell - \imath_1^+ + n - 2 + 1 + Z_{(\ell, \jmath_1)}(G)$$

simple zeros. Hence,

$$\ell - \imath_1^+ + n - 1 + Z_{(\ell, \jmath_1)}(G) \leqslant m_1 - \tau - 1,$$

since $G - \varepsilon Q^\tau$ is in the $(m_1 - \tau)$-dimensional WT-space $\mathscr{S}_1^{0,\tau}$. Thus,

$$Z_{(\ell, \jmath_1)}(G) \leqslant m_1 - \tau - 1 - (\ell - \imath_1^+ + n - 1) = \jmath_1^- - \ell - \tau - 1. \qquad \square$$

4.3. *Proof that the spaces $\mathscr{S}_\Pi$ and $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfy Property A*

We are finally in a position to prove Theorem 4.12.

Proof of Theorem 4.12. Recall that the case $n = 2$ was proved in Theorem 4.20. So we suppose $n \geqslant 3$. From Lemmas 4.13 and 4.14, the proof of Theorem 4.12 is reduced to the case in which $\Gamma = \mathbb{Z}$ and u has at most one maximal zero interval.

Assume first that u has no zero interval, and let Ω be the set of simple zeros of u^*. Lemma 4.15 shows that if there exist indexes $i, j \in \mathbb{Z}$ such that $\Omega_{(i,j)} < j - i - n + 1$, then Theorem 4.12 holds. Hence, we suppose $\Omega_{(i,j)} \geqslant j - i - n + 1$ for each $i < j$. Thus, we are assuming that $\{u, \Omega\}$ is a reference pair.

Claim 1. *Theorem* 4.12 *holds when* $\{u, \Omega\}$ *is a reference pair.*

From Lemma 4.49 it follows that in order to prove Theorem 4.12 in this case, it is sufficient to demonstrate that there exists an $H_0 \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$ such that Ω is an alternating set for $\Psi(H_0)$, where ρ_0 and τ_0 satisfy the conditions in (13). We now prove the existence of such a spline $H_0 \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$. Taking into account (13), we apply Property [JKZ] in the $(m_1 - \tau_0 - \rho_0)$-dimensional WT-space $\mathscr{S}_1^{\rho_0,\tau_0}$ to obtain an $H \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$ for which $\Omega \cap (x_{\imath_1}, x_{\jmath_1})$ is an alternating set. There are two cases to consider:

First Case. The spline H has no zero interval. In this case take $H_0 := H$. Then from Lemma 4.50, Ω is an alternating set for $\Psi(H_0)$. Thus, Claim 1 is proved in this case.

Second Case. The spline H has a zero interval. In this case, according to Remark 4.51, we consider an $\bar{H} \in \mathscr{S}_1^{\rho_0,\tau_0} \backslash \{0\}$ for which $\Omega \cap (x_{\iota_1}, x_{J_1})$ is an alternating set, and such that $\bar{H}$ coincides with H on $[x_{\ell_0}, x_{J_1}]$, and $[x_{\iota_1}, x_{\ell_0}]$ is a maximal zero interval of $\bar{H}$, $\iota_1 < \ell_0 < J_1^-$. Then $\bar{H}$ is actually in $\mathscr{T}_{\ell_0}^{\tau_0} \backslash \{0\} \subset \mathscr{S}_1^{\rho_0,\tau_0}$. Moreover, using Lemma 4.47 we deduce that $(a, x_{\ell_0}]$ is the only maximal zero interval of $\bar{h} := \Psi(\bar{H})$, whence $\tau_0 < \tau^*$.

Assume that $\bar{H}$ has $J_1^- - \ell_0 - \tau_0 - 1$ sign changes. Then applying Lemma 4.54(a) to $\bar{H} \in \mathscr{T}_{\ell_0}^{\tau_0} \backslash \{0\}$ we see that $\bar{h}$ has neither double zeros nor zero intervals in (x_{ℓ_0}, b), and $\{\bar{h} = 0\} \cap [x_{J_1}, b) = \Omega \cap [x_{J_1}, b)$. So, taking into account that $\Omega \cap (x_{\iota_1}, x_{J_1})$ is an alternating set for $\bar{H}$, we conclude that Ω is an alternating set for $\bar{h}$. Then in this situation take $H_0 = \bar{H}$, and Claim 1 is proved.

Assume now that $\bar{H}$ does not have $J_1^- - \ell_0 - \tau_0 - 1$ sign changes. Under this condition, we will show the existence of an $H_0 \in \mathscr{T}_{\ell_0}^{\tau_0'} \backslash \{0\}$, $\tau_0 < \tau_0' < \tau^*$, with $J_1^- - \ell_0 - \tau_0' - 1$ sign changes. In this manner, we will be able to apply again Lemma 4.54, this time to $H_0 \in \mathscr{T}_{\ell_0}^{\tau_0'} \subset \mathscr{S}_1^{\rho_0,\tau_0}$, to complete the proof of the claim. Note that $\bar{H}$ has $\Omega_{(\ell_0,J_1)}$ sign changes on (x_{ℓ_0}, x_{J_1}), since $\bar{H}$ has no zero interval in $[x_{\ell_0}, x_{J_1}]$. Then, as $\mathscr{T}_{\ell_0}^{\tau_0}$ is a $(J_1^- - \ell_0 - \tau_0)$-dimensional WT-space, we are assuming that

$$\Omega_{(\ell_0,J_1)} < J_1^- - \ell_0 - \tau_0 - 1.$$

We now prove that $J_1^- - \ell_0 - \tau^* - 1 < \Omega_{(\ell_0,J_1)}$. Suppose to the contrary that $(0 \leqslant)$ $\Omega_{(\ell_0,J_1)} \leqslant J_1^- - \ell_0 - \tau^* - 1$. Therefore, after applying [SSS] if $\Omega_{(\ell_0,J_1)} < J_1^- - \ell_0 - \tau^* - 1$, we use Property [JKZ] in an $(\Omega_{(\ell_0,J_1)} + 1)$-dimensional WT-subspace of $\mathscr{T}_{\ell_0}^{\tau^*}$ to obtain a nontrivial spline in $\mathscr{T}_{\ell_0}^{\tau^*} \subset \mathscr{S}_1^{\rho^*,\tau^*}$ for which $\Omega \cap (x_{\iota_1}, x_{J_1})$ is an alternating set. This contradicts Lemma 4.47. Therefore, $J_1^- - \ell_0 - \tau^* - 1 < \Omega_{(\ell_0,J_1)}$, and so we get $J_1^- - \ell_0 - \tau^* - 1 < \Omega_{(\ell_0,J_1)} < J_1^- - \ell_0 - \tau_0 - 1$. Hence, there exists an integer τ_0' such that

$$\Omega_{(\ell_0,J_1)} = J_1^- - \ell_0 - \tau_0' - 1,$$

with $\tau_0 < \tau_0' < \tau^*$. Then we can use Property [JKZ] in the $(J_1^- - \ell_0 - \tau_0')$-dimensional WT-space $\mathscr{T}_{\ell_0}^{\tau_0'}$ to obtain an $H_0 \in \mathscr{T}_{\ell_0}^{\tau_0'} \backslash \{0\}$ for which $\Omega \cap (x_{\iota_1}, x_{J_1})$ is an alternating set.

Assertion. *The spline H_0 has no zero interval in $[x_{\ell_0}, x_{J_1}]$.*

Assume, contrary to our assertion, that H_0 has a zero interval in $[x_{\ell_0}, x_{J_1}]$. By Lemma 4.47, H_0 cannot have two maximal zero intervals, since $\Omega \cap (x_{\iota_1}, x_{J_1})$ is an alternating set for H_0. Hence, we are assuming that there exists an integer ℓ_1 satisfying $\ell_0 + 1 \leqslant \ell_1 \leqslant J_1^- - 1$ and such that H_0 is in $\mathscr{T}_{\ell_1}^{\tau_0'}$ without zero interval in $[x_{\ell_1}, x_{J_1}]$. Consider again the spline $\bar{H}$, in $\mathscr{T}_{\ell_0}^{\tau_0}$. As $\bar{H}$ has no zero intervals in $[x_{\ell_0}, x_{J_1}]$, the restriction of $\bar{H}$ to $[x_{\ell_0}, x_{\ell_1}]$ is in $\mathscr{S}^+_{\ell_0,\ell_1}$ and has no zero interval. Then

$$\Omega_{(\ell_0,\ell_1]} \leqslant Z_{(\ell_0,\ell_1]}(\bar{H}) \leqslant \ell_1 - \ell_0 - 1,$$

where the last inequality is due to Lemma 4.11(b). Therefore,

$$\begin{aligned}\Omega_{(\ell_1,J_1)} &= \Omega_{(\ell_0,J_1)} - \Omega_{(\ell_0,\ell_1]} \\ &\geqslant J_1^- - \ell_0 - \tau_0' - 1 - (\ell_1 - \ell_0 - 1) = J_1^- - \ell_1 - \tau_0'.\end{aligned}$$

As $\Omega \cap (x_{\ell_1}, x_{J_1})$ is an alternating set for H_0, and H_0 has no zero interval in (x_{ℓ_1}, x_{J_1}), we conclude that H_0, in $\mathscr{T}^{\tau_0'}_{\ell_1}$, has at least $J_1^- - \ell_1 - \tau_0'$ sign changes. This contradicts Theorem 4.55 applied to $\mathscr{T}^{\tau_0'}_{\ell_1}$. So the assertion is proved.

Taking into account that $\Omega \cap (x_{\ell_0}, x_{J_1})$ is an alternating set for $H_0 \in \mathscr{T}^{\tau_0'}_{\ell_0}$, that H_0 has no zero interval in $[x_{\ell_0}, x_{J_1}]$ and that $\Omega_{(\ell_0,J_1)} = J_1^- - \ell_0 - \tau_0' - 1$, we deduce that H_0 has $J_1 - \ell_0 - \tau_0' - 1$ sign changes. Then from Lemma 4.54(a) applied to $H_0 \in \mathscr{T}^{\tau_0'}_{\ell_0} \backslash \{0\}$ we conclude that $h_0 := \Psi(H_0)$ has neither double zeros nor zero intervals in $[x_{\ell_0}, b)$, and

$$\{h_0 = 0\} \cap [x_{\ell_0}, b) = \Omega \cap [x_{\ell_0}, b).$$

Accordingly, under the present assumption on $\bar{H}$ we have also proved that there exists a nontrivial spline H_0 in $\mathscr{T}^{\tau_0'}_{\ell_0} \subset \mathscr{S}^{\rho_0,\tau_0}_1$ such that Ω is an alternating set for h_0 $(= \Psi(H_0))$. This completes the proof of Claim 1.

Observe that it only remains to prove the theorem for the case in which $\Gamma = \mathbb{Z}$ and u has exactly one maximal zero interval. Thus, the following claim completes the proof of Theorem 4.12.

Claim 2. *Theorem* 4.12 *holds when* $\Gamma = \mathbb{Z}$ *and u has exactly one maximal zero interval.*

It is obvious that by arguments of symmetry we can assume that the maximal zero interval of u is not of the form $[x_j, b)$ for any $j \in \mathbb{Z}$. Under this assumption, $[x_i, x_{i+1}]$ is a maximal zero interval of $u|_{[x_i,b)}$ for some $i \in \mathbb{Z}$. We will next find an h_0 that makes Theorem 4.12 to hold and that it vanishes on $(a, x_{i+1}]$ independently of the form of u on (a, x_i). Therefore, we can suppose without loss of generality that $(a, x_0]$ is a maximal zero interval of u.

For $i \in -\mathbb{N}$, take one point z_i in each (x_{i-1}, x_i). It is easily seen that there exists a $u_0 \in \mathscr{S}_\Pi$ without zero intervals, and without double zeros in $(a, x_{-1}]$, and such that $u_0 = u$ on $[x_0, b)$ and $u_0(z_i) = 0$, all $i \in -\mathbb{N}$. It is easy to check that $z_{-1}, z_{-2}, \ldots$ are the only zeros of u_0 in (a, x_0). In particular, $\sigma(\imath_1) = 0$ and $\imath_1^+ = \imath_1$. Set $u_0^* = u^*$ on $[x_0, b)$, and on (a, x_0) take $u_0^* = u_0$, or $u_0^* = -u_0$, where the sign is chosen so that x_0 becomes a simple zero of u_0^*. Let $\Omega := \{z \in J$: z is a simple zero of $u_0^*\}$. We assume

$$\mathrm{Card}(\Omega \cap (x_i, x_j)) \geqslant j - i - n + 1 \quad \text{for all } 0 \leqslant i < j,$$

since otherwise we would use Property A in $\mathscr{S}^0_{i,j}$ to obtain a straightforward proof of Theorem 4.12. Under these conditions, $\{u_0, \Omega\}$ becomes a reference pair, and therefore we will next apply Claim 1 to that reference pair. As $u_0 = u$ on $[x_0, b)$ and u does not go to 0 to the right, it follows that both u and u_0 have the same right level as $v_0^{0,\tau(u_0)}$ for some integer $\tau(u_0)$ satisfying $0 \leqslant \tau(u_0) < \tau^*$. Let U_0 and U be the restrictions of u_0 and u to $[x_{\imath_1}, x_{\jmath_1}]$, respectively. Then $U_0 \in \mathscr{S}_1^{\rho(u_0),\tau(u_0)} \setminus \{0\}$ and $U \in \mathscr{T}_0^{\tau(u_0)} \setminus \{0\}$ (in fact, $\rho(u_0) = 0$ but this does not matter). As U has no zero intervals in $[x_0, x_{\jmath_1}]$, it follows that $\Omega_{(0,\jmath_1)} \leqslant Z_{(0,\jmath_1)}(U)$. Then from Lemma 4.56 applied to U, we get

$$\Omega_{(0,\jmath_1)} \leqslant Z_{(0,\jmath_1)}(U) \leqslant \jmath_1^- - \tau(u_0) - 1.$$

On the other hand, it is easy to see that $\Omega_{(\imath_1,\jmath_1)} = -\imath_1 + \Omega_{(0,\jmath_1)}$. We now choose ρ_0 and τ_0 satisfying the second condition in (13). Take $\rho_0 = \rho^*$. Due to the location of the points z_i in $(a, x_{\imath_1}]$ it follows easily that $\rho^* = n - 1$. From Lemma 4.47, $m_1 - \tau^* - \rho^* - 1 < \Omega_{(\imath_1,\jmath_1)}$. Thus, pick an integer $\tau_0 < \tau^*$ such that $m_1 - \tau_0 - \rho^* - 1 = \Omega_{(\imath_1,\jmath_1)}$. Then we have

$$\jmath_1^- - \imath_1 - \tau_0 - 1 = \jmath_1^- - \imath_1 + n - 1 - \tau_0 - \rho^* - 1 = -\imath_1 + \Omega_{(0,\jmath_1)}.$$

Therefore, $\jmath_1^- - \tau_0 - 1 = \Omega_{(0,\jmath_1)}$. Thus $\tau_0 \geqslant \tau(u_0)$, and obviously, $\rho^* \geqslant \rho(u_0)$. Accordingly, (13) holds with $\rho_0 = \rho^*$, and τ_0 (and with u_0 in place of u). Now, as $\jmath_1^- - \tau_0 - 1 = \Omega_{(0,\jmath_1)}$ and $\mathscr{T}_0^{\tau_0}$ is a $(\jmath_1^- - \tau_0)$-dimensional WT-space, using Property [JKZ] we see that there exists a nontrivial H in that space for which $\Omega \cap (0, x_{\jmath_1})$ is an alternating set. Hence, $\Omega \cap (x_{\imath_1}, x_{\jmath_1})$ is also an alternating set for $H \in \mathscr{T}_0^{\tau_0} \subset \mathscr{S}_1^{\rho^*,\tau_0}$. At this point we apply the second case in Claim 1 to obtain an h_0 satisfying (a)–(c) of Theorem 4.12 for the function u_0^*. Observe that $h_0 = 0$ on $(a, x_0]$. Then it is immediate to see that h_0 also satisfies (a)–(c) of Theorem 4.12 for the function u^*. This completes the proof of Claim 2. □

Theorem 4.12 shows that $\mathscr{S}_\Pi \cap \mathscr{L}_\phi$ satisfies Property A. From Theorem 3.3 we therefore conclude the following result.

Uniqueness theorem. *Let ϕ be a convex function defined on $[0, \infty)$, with $\phi(0) = 0$ and $\phi(y) > 0$ for $y > 0$, and also assume that ϕ satisfies Property Δ_2. Let J denote an open interval and let f be a continuous function in $\mathscr{L}_\phi$. Then there exists a unique $g_0 \in \mathscr{S}_\Pi$ such that*

$$\int_J \phi(|f - g_0|) \leqslant \int_J \phi(|f - g|) \quad \textit{for every } g \in \mathscr{S}_\Pi.$$

Example. For $J = (-\infty, +\infty)$ and $n = 3$ we will construct a u_0 in $\mathscr{S}_\Pi \setminus L_1$ with infinitely many isolated zeros. Applying to u_0 the procedure expounded in the proof of Theorem 4.12 we will obtain an $h_0 \in \mathscr{S}_\Pi$, also with infinitely many isolated zeros, and with a lower right and left level than u_0. We will prove that h_0 is in L_1, which shows the effective action of the levels. We emphasize that the proof of existence of this kind of splines in L_1 is an interesting application of the theory of levels.

Let $J = (-\infty, +\infty)$ and $n = 3$, and let $\Pi = \{x_i\}_{i\in\mathbb{Z}}$, where $x_i = i$ for all $i\in\mathbb{Z}$. For every integer $i\geqslant 1$, let us denote by z_i the middle point of the interval $[x_i, x_{i+1}]$, and let $z_{-i} = -z_i$ and $z_0 = x_0$. It is easy to see that the restriction of the function x^2 to $[-1, 1]$ can be extended to an even function $u_0\in\mathscr{S}_\Pi$ in such a way that the only zeros of u_0 are the points $\{z_i\}_{i\in\mathbb{Z}}$ and where z_0 is the only double zero of u_0. Set $\Omega := \{z_i\}_{i\in\mathbb{Z}}$. Then $\{u_0, \Omega\}$ is a reference pair. For each $v\in\mathbb{N}$, let $\imath_v := -v-2$ and $\jmath_v := v+2$. Therefore, the sequences $\{\imath_v\}_{v\in\mathbb{N}}$ and $\{\jmath_v\}_{v\in\mathbb{N}}$ satisfy (2)–(5), and so we can apply the theory developed in the proof of Theorem 4.12. Observe that for all $v\in\mathbb{N}$, the $(6+2v)$-dimensional WT-space $\mathscr{S}_v$ coincides with $\mathscr{S}_{-v-2,v+2}$, since $u_0(x_{\imath_v})\neq 0$ and $u_0(x_{\jmath_v})\neq 0$. Moreover, it is not difficult to see that there are nine WT-spaces $\mathscr{S}_1^{\rho,\tau}$, $0\leqslant\tau\leqslant\tau^* = 2$ and $0\leqslant\rho\leqslant\rho^* = 2$, being $\mathscr{S}_1^{0,0} = \mathscr{S}_1 = \mathscr{S}_{-3,3}$, $\mathscr{S}_1^{0,2} = \mathscr{S}_{-3,3}^-$, $\mathscr{S}_1^{2,0} = \mathscr{S}_{-3,3}^+$ and $\mathscr{S}_1^{2,2} = \mathscr{S}_{-3,3}^0$.

We claim that u_0 has the same right and left levels as $v_0^{0,0}$ $(= \Psi(V_0))$, where V_0 is in the basis for $\mathscr{S}_1^{0,0}$ introduced in Remark 4.29. Observe that V_0 is an odd spline—and hence $v_0^{0,0}$ as well—if we choose, as we do, a symmetric set for its seven zeros y_k. Define $G\in\mathscr{T}_0^0$ as $G = 0$ on $[-3, 0]$ and $G(x) = U_0(x)$ for all $x\in[0, 3]$, where U_0 is the restriction of u_0 to $[-3, 3]$. Since G has two sign changes, G cannot be in the 2-dimensional WT-space $\mathscr{T}_0^1$. Then $\Psi(G)$ has the same right level as $v_0^{0,0}$, and the same fact is valid for u_0 because $u_0 = \Psi(G)$ on $[0, +\infty)$. By arguments of symmetry we deduce that u_0 has the same left level as $v_0^{0,0}$, and this proves the claim.

We now see that $u_0\notin L_1$. To do this, consider the broken line q defined on $(-\infty, +\infty)$ in such a way that $q(x) = |x|$ for all $x\in[-1, 1]$ and $q(z_i) = 0$ for every $i\in\mathbb{Z}$. It is not difficult to see that $|q| < |u_0|$ on $((-\infty, +\infty)\backslash[-1, 1])\backslash\Omega$. Then $\int_{-\infty}^{+\infty} |u_0| = \infty$, since $\int_1^{+\infty} |q| = \infty$. Thus, we conclude that $u_0\notin L_1$.

We have just seen that u_0 has the same right and left level as $v_0^{0,0}$ and that $u_0\notin L_1$. So, in order to show the effective action of levels on $\Psi(\mathscr{S}_1)$, our aim is now to prove that if $H\in\mathscr{S}_1^{1,1}$, then $\Psi(H)\in L_1$. As $\mathrm{Card}(\Omega\cap(-3, 3)) = 5$, we can apply Property [JKZ] in the 6-dimensional WT-space $\mathscr{S}_1^{1,1}$ to obtain an $H_0\in\mathscr{S}_1^{1,1}\backslash\{0\}$ for which $\Omega\cap(-3, 3)$ is an alternating set. Note that H_0 has no zero interval, since otherwise either H_0 or $H_0(-x)$ would be in the 2-dimensional WT-space $\mathscr{T}_0^1$, with two sign changes, which is not possible. Therefore, H_0 has five sign changes. By the location of the points z_i we see that $h_0 := \Psi(H_0)$ does not go to 0 to the right nor to the left. We can thus apply (b_4) of Theorem 4.36 to $H_0\in\mathscr{S}_1^{1,1}$. It follows that the points in Ω are the only (simple) zeros of h_0. Observe that H_0 is uniquely determined up to a multiplicity constant. In particular, this implies that H_0 is an odd function, and hence h_0 as well. We now prove the following assertion.

Assertion 1. *The broken line h_0' has a simple zero in each (x_i, x_{i+1}), all $i\in\mathbb{Z}$.*

From (b_4) of Theorem 4.36 applied to $H_0\in\mathscr{S}_1^{1,1}$, h_0' has at least one zero in $[x_i, x_{i+1}]$, all $i\in\mathbb{Z}$. From this condition and by the location of the zeros of h_0, we see that the broken line h_0' has no zero interval. Therefore, h_0' has exactly one zero in each component $[x_i, x_{i+1}]$, $i \in \mathbb{Z}$. Hence, to prove the assertion it remains to show that h_0' has no zero at the knots x_i for any $i \in \mathbb{Z}$.

Suppose first that h_0' has a double zero at a knot x_l for some $l \in \mathbb{Z}$. If $l > 0$, then h_0' has at most $l-2$ sign changes on (x_0, x_{l+1}). Therefore, applying Lemma 4.8 to the restriction of h_0' to (x_0, x_{l+1}) we obtain a contradiction, since h_0 has l simple zeros in $[x_0, x_{l+1}]$. Obviously, the same argument is valid when $l < 0$. Thus, h_0' has no double zero at the knots x_i for any $i \neq 0$. Finally, if h_0' has a double zero at x_0, then it is not difficult to see that there exists a constant λ such that $h_0 = \lambda u_0$ on $(x_0, +\infty)$ and $h_0 = -\lambda u_0$ on $(-\infty, x_0)$. This contradicts that $H_0 \in \mathscr{S}_1^{1,1}$, since $U_0 \notin \mathscr{S}_1^{1,1}$.

Assume now that h_0' has a simple zero at a knot $x_{i'}$ for some $i' \in \mathbb{Z}$. As h_0 is an odd function, we see that $i' \neq 0$ and that h_0' has also a simple zero at the knot $x_{-i'}$. Suppose $i' > 0$. Thus, for all ε sufficiently small the following two conditions hold:

- The broken line $(h_0 - \varepsilon v_0^{0,0})'$ has at most $2j-2$ sign changes on (x_{-j}, x_j) for every $j > i'$.
- The spline $h_0 - \varepsilon v_0^{0,0}$ has at least five sign changes on $(-3, 3)$.

Note that for all ε, the spline $h_0 - \varepsilon v_0^{0,0}$ has no zero interval in $(3, +\infty)$. Otherwise we would conclude that $h_0 - \varepsilon v_0^{0,0} = 0$ on $(3, +\infty)$, which is a contradiction because the right levels of h_0 and $v_0^{0,0}$ are different. In the same way $h_0 - \varepsilon v_0^{0,0}$ has no zero interval in $(-\infty, -3)$. Take an integer $r > \max\{i', 3\}$. Then since $(h_0 - \varepsilon v_0^{0,0})(z_j) = 0$ for each $j \geqslant 3$, using the second condition above we deduce that for all ε sufficiently small, $Z^2_{[x_{-r}, x_r]}(h_0 - \varepsilon v_0^{0,0}) \geqslant 5 + \mathrm{Card}(\Omega \cap ([x_{-r}, x_r] \backslash (-3, 3)) = 5 + 2r - 6$. So taking into account the first of the two conditions above and using Lemma 4.8 we get $Z^2_{[x_{-r}, x_r]}(h_0 - \varepsilon v_0^{0,0}) = 5 + \mathrm{Card}(\Omega \cap ([x_{-r}, x_r] \backslash (-3, 3))$. We conclude that for all ε sufficiently small, $h_0 - \varepsilon v_0^{0,0}$ has no double zero, and

$$\{h_0 - \varepsilon v_0^{0,0} = 0\} \cap (J \backslash (-3, 3)) = \Omega \cap (J \backslash (-3, 3)).$$

On the other hand, applying (b$_4$) of Theorem 4.36 to $H_0 \in \mathscr{S}_1^{1,1}$ and to $V_0^{0,0} \in \mathscr{S}_1^{0,0}$ we deduce that h_0 and $v_0^{0,0}$ have no double zero and

$$\{h_0 = 0\} \cap (J \backslash (-3, 3)) = \{v_0^{0,0} = 0\} \cap (J \backslash (-3, 3)) = \Omega \cap (J \backslash (-3, 3)).$$

Accordingly, apply Lemma 4.28(c) to h_0 and $v_0^{0,0}$ to deduce that h_0 does not have a lower right or left level than $v_0^{0,0}$. This contradicts that $H_0 \in \mathscr{S}_1^{1,1}$. Thus, h_0' has no simple zero at the knots x_i for any $i \in \mathbb{Z}$, which completes the proof of the assertion.

In accordance with Assertion 1, for any integer $i \geqslant 0$ let c_i be the simple zero of h_0' in the interval (x_i, x_{i+1}).

Assertion 2. *For each integer* $i \geqslant 0$,

$$\frac{z_i + x_{i+1}}{2} < c_i < x_{i+1}.$$

To prove the assertion, observe first that by definition of c_i, we have $c_i < x_{i+1}$ for each integer $i \geqslant 0$. Recall also that the points in Ω are the only (simple) zeros of h_0. Without loss of generality, assume $h_0 > 0$ on $(x_0, x_1]$. Therefore, h_0' is decreasing in the interval $[x_0, x_1]$, since $h_0'(0) > 0$ and h_0' has a simple zero c_0 in (x_0, x_1). Moreover, $c_0 > (x_0 + x_1)/2$. Otherwise h_0 would have a zero in $(x_0, x_1]$. This proves the assertion for $i = 0$ because $z_0 = x_0$. As $h_0 > 0$ on (x_0, z_1) and $h_0' < 0$ on (c_0, c_1), we deduce that $h_0' < 0$ on (x_1, z_1). Furthermore, $h_0'(z_1) < 0$, since z_1 is a simple zero of h_0. Therefore, $c_1 > z_1$. Suppose now $c_1 \leqslant (z_1 + x_2)/2$. As $h_0(z_1) = 0$, we get $h_0(x) = \int_{z_1}^{x} h_0'$ for all $x \in [z_1, x_2]$. Then we see that h_0 has a zero at a point in $(z_1, x_2]$, which is a contradiction. Thus, $c_1 > (z_1 + x_2)/2$. Now, as $h_0 < 0$ on (z_1, z_2) and $h_0' > 0$ on (c_1, c_2), in the same manner we can see that $c_2 > (z_2 + x_3)/2$. The proof of the inequality $\frac{z_i + x_{i+1}}{2} < c_i$ now falls into a recurrent procedure, that allows us to complete the proof of the assertion by induction on i.

From Assertion 2 with $i = 0$ it follows that $|h_0'(x_1)| < |h_0'(x_0)|$, and with $i = 1$ we see that

$$|h_0'(z_1)| < \tfrac{1}{2}|h_0'(x_1)| \quad \text{and} \quad |h_0'(x_2)| < \tfrac{1}{3}|h_0'(x_1)|.$$

Therefore,

$$|h_0'(z_1)| < \tfrac{1}{2}|h_0'(x_0)| \quad \text{and} \quad |h_0'(x_2)| < \tfrac{1}{3}|h_0'(x_0)|.$$

In general, as $(z_j + x_{j+1})/2 < c_j < x_{j+1}$ for each integer $j \geqslant 1$, we deduce that

$$|h_0'(z_j)| < \tfrac{1}{2}|h_0'(x_j)| \quad \text{and} \quad |h_0'(x_{j+1})| < \tfrac{1}{3}|h_0'(x_j)|.$$

Thus, for each integer $j \geqslant 1$,

$$|h_0'(z_j)| < \tfrac{1}{2}\left(\tfrac{1}{3}\right)^{j-1}|h_0'(x_0)|.$$

On the other hand, note that for each $j \geqslant 1$, $|h_0'(z_j)| \geqslant |h_0'(x)|$ for every $x \in [z_j, z_{j+1}]$. Hence, for all $x \in [z_j, z_{j+1})$,

$$|h_0(x)| \leqslant \int_{z_j}^{x} |h_0'(z_j)| < \int_{z_j}^{z_{j+1}} \frac{1}{2}\left(\frac{1}{3}\right)^{j-1} |h_0'(x_0)| = \frac{1}{2}\left(\frac{1}{3}\right)^{j-1} |h_0'(x_0)|.$$

Accordingly,

$$\begin{aligned}\int_{z_1}^{+\infty} |h_0| = \textstyle\sum_{j=1}^{\infty} \displaystyle\int_{z_j}^{z_{j+1}} |h_0| &< \sum_{j=1}^{\infty} \int_{z_j}^{z_{j+1}} \frac{1}{2}\left(\frac{1}{3}\right)^{j-1} |h_0'(x_0)| \\ &= \frac{1}{2}\sum_{j=1}^{\infty} \left(\frac{1}{3}\right)^{j-1} |h_0'(x_0)| = \frac{3}{4}|h_0'(x_0)|.\end{aligned}$$

In this way, taking into account that h_0 is an odd function, we get

$$\int_{-\infty}^{+\infty} |h_0| = \int_{-\infty}^{-z_1} |h_0| + \int_{-z_1}^{z_1} |h_0| + \int_{z_1}^{+\infty} |h_0| < \frac{3}{4}|h_0'(x_0)| + \int_{-z_1}^{z_1} |h_0| + \frac{3}{4}|h_0'(x_0)| < \infty,$$

whence $h_0 \in L_1$.

Finally, applying Theorem 4.46, with H_0 in place of G, we conclude that any H in $\mathscr{S}_1^{1,1}$ has the property that $\Psi(H)$ is in L_1.

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